Arms Race Model

Robert Joseph
Maninder Dhaliwal
Manaal Basha

UNIVERSITY OF
ALBERTA

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Robert Joseph
Manaal Basha
Maninder Dhaliwal
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Abstract

This project investigates the behaviours of a nation when engaged in an arms race; we wanted to know the kinds of things that could change the course of the race. Our approach was centered around the relationships between nations. This allowed us to add and manipulate factors that directly affect the economic allocations a nation can make to fund its military production. We then implicated the changes in a nation's economy as the consequences of their perceivable expenditure, leading to a series of optimizations that would maximize production.

Contents

Chapter 1

Mutual Fear Model

1.1 Introduction

The demonstration of political power has been at the forefront of the global market since economically influential nations began investing in their own ideals. The struggle for power at an international scale has been realized in many forms of competition, whether it be sports or technological advances, nations and those who make decisions on their peoples behalf are influenced by their necessity to express dominance. A particularly formidable form of competition is of concern to us in the form of an Arms Race that drives nations to develop arsenals that are capable of mass destruction. In this paper we will utilize the understanding that the nations in an arms race are driven by the fear of losing control of the lead on not only military power, but the ability to withstand their opposition should any hostility be acted upon. This fear is related to the expenditures a nation pursues in their reactions to one another and we will look to expand on the mitigating and compounding factors and that relate to the economical states of the nations involved.

We begin with the consideration of two economically competing nations which we will call Purple and Green, both desire peace and hope to avoid war, yet they are not pacifistic. They will not go out of their way to display aggression, but they will not sit idly by if their country is attacked. They believe in self-defense and will fight to protect their nation and their way of life. Both nations feel that the maintenance of a large army and the stockpiling of weapons are purely "defensive" gestures when they do it, but at least somewhat "offensive" when the other side does it. Since the two nations compete in this way, there is an underlying sense of "mutual fear." The more one nation arms, the more the other nation is spurred to arm.

Let $x(t)$, and $y(t)$ represent the yearly rates of armament expenditures of the two nations in some standardized monetary unit. To develop a model of mutual fear, we assume that each country increases or decreases their armament expenditures in response to the expenditure levels of the other.

1.1.1 Assumptions

The simplest assumption is that each nation's rate is directly proportional to the expenditure of the other nation, that is,

$$\frac{dx}{dt} = ay$$
$$\frac{dy}{dt} = bx$$

where a and b are positive constants.

1.2 Solution

Hyperbola

This a simple separable linear equation

$$\implies \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{bx}{ay}$$
$$\implies ay \cdot dy = bx \cdot dx$$
$$\implies \frac{ay^2}{2} = \frac{bx^2}{2} + C$$

Applying the initial conditions $x(0) = x_0$ and $y(0) = y_0$ that are representative of the initial rates, we have

$$\implies \frac{ay^2}{2} - \frac{bx^2}{2} = \frac{a(y_0)^2}{2} - \frac{b(x_0)^2}{2}$$

and dividing the above by $\frac{2}{ab}$ yields

$$\implies \frac{y^2}{(\sqrt{(b)}^2)} - \frac{x^2}{(\sqrt{(a)}^2)} = C$$

.

This leads to a desirable outcome of the relationship being on a Hyperbola with a centre at the origin $(0,0)$.

We now expand on the above by letting

$$\frac{(x_0)^2}{a} - \frac{(y_0)^2}{b} = C$$

where we can fix C and a,b and better understand the implications of the above.

Let us take $C = 1, a = 1$ and $b = 1$ to we get $x^2 - y^2 = 1$. With the help of desmos (an online graphing tool) we visualize this hyperbolic equation as it depends on the values of C. The interactive tool illustrates that the hyperbola switches and flips 90 degrees if the constant C turns negative. Our understanding is developed in the following cases:

Case 1

When $C > 0$, the graph implies that if nation x increases its arms expenditure, nation y will also increase or decrease but never cross the expenditure of x. If x decreases its arms expenditure then we have an increase/decrease in y but it will always be more than the decrease of x.

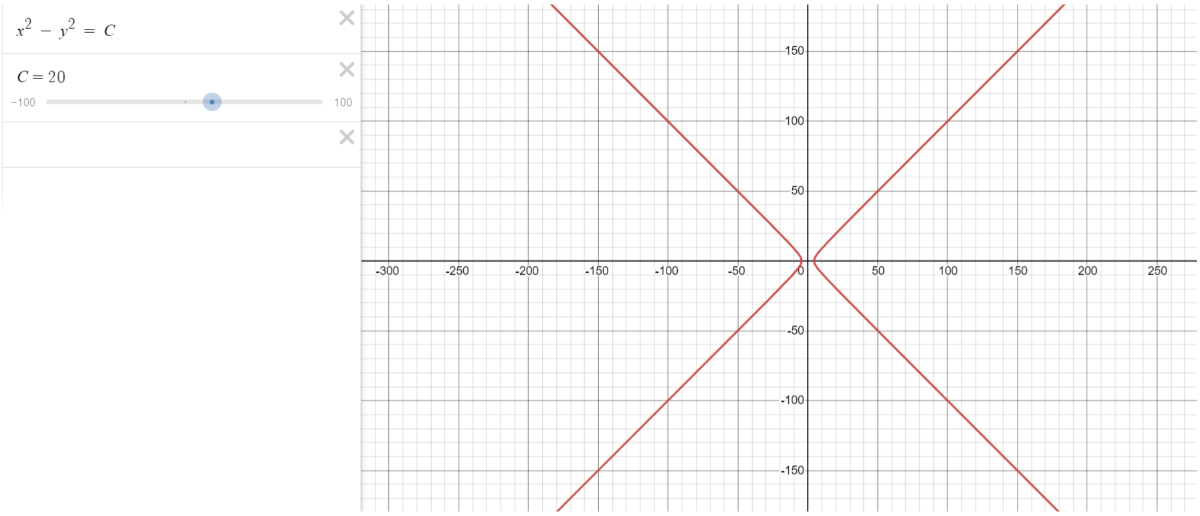


Figure 1.1

Case 2

When $C < 0$, we see the roles are reversed for both the cases above and hence similarly this is proved.

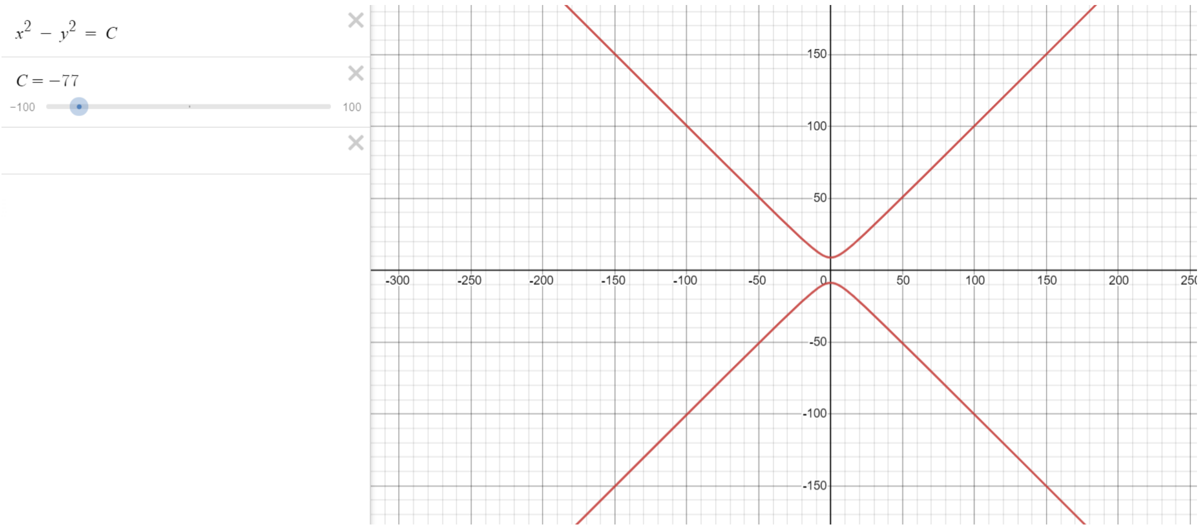


Figure 1.2

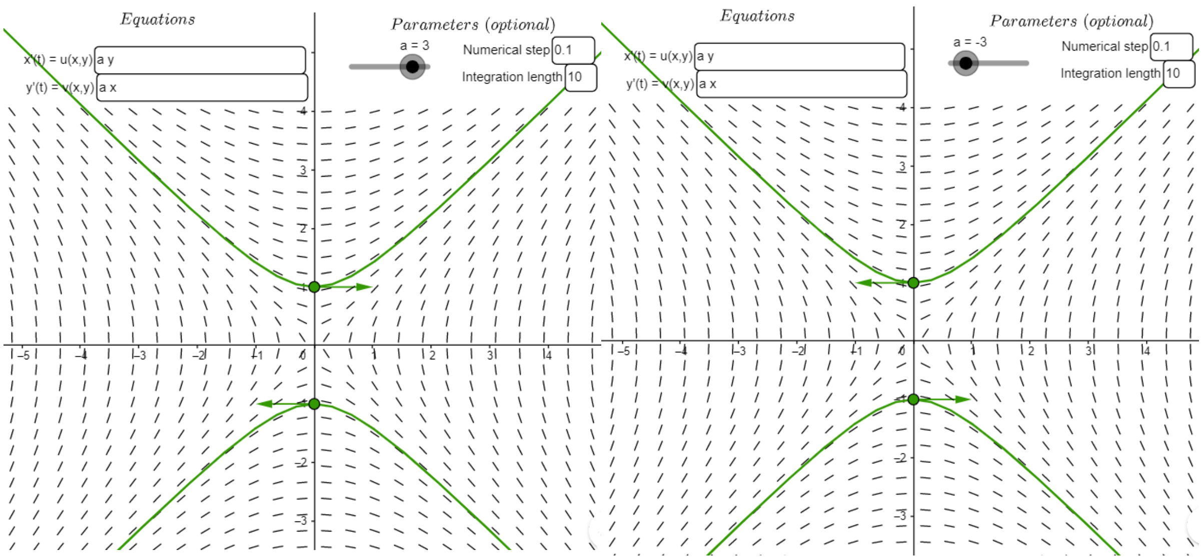
1.2.1 Equilibrium Values

We now look to find the equilibrium values that are given by $(x, y) = (0, 0)$, which are done by solving for $(0, 0) = (ay, bx)$.

Jacobian

We find the following Jacobian for this system of ODEs,

$$J_{(}x,y) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$



(a) $a,b>0$

(b) $a,b<0$

Figure 1.3: Mutual Fear Model - Phase Potraits

which illuminates the $\det(J)=-ab$ and $tr(J)=0$ where $a,b\in\mathbb{R}^{+}$.

This is clearly an unstable node and hence following the phase plane analysis results in an unstable system. We can also plot the phase portraits as shown in Figure's 1.3 and 1.4.

1.2.2 System of ODE Solution

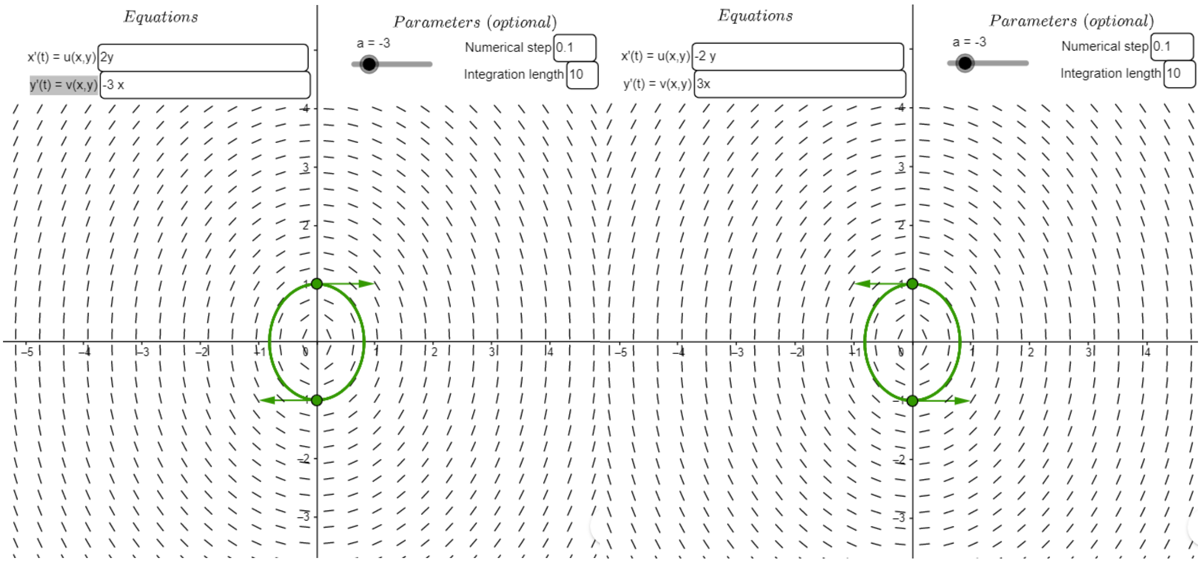
We can also plot the system of ODE solutions as shown in Figure 1.5.

1.2.3 Conclusion

Hence combining both of these, yields in a very unstable system alternating between an increase/decrease in arms race.

1.3 Critique

The mutual fear model produced a runaway arms race with unlimited expenditures which we determined to be an unreasonable assumption.



(a) $a > 0, b < 0$ *(b)* $a < 0, b > 0$

Figure 1.4: Mutual Fear Model - Phase Potraits

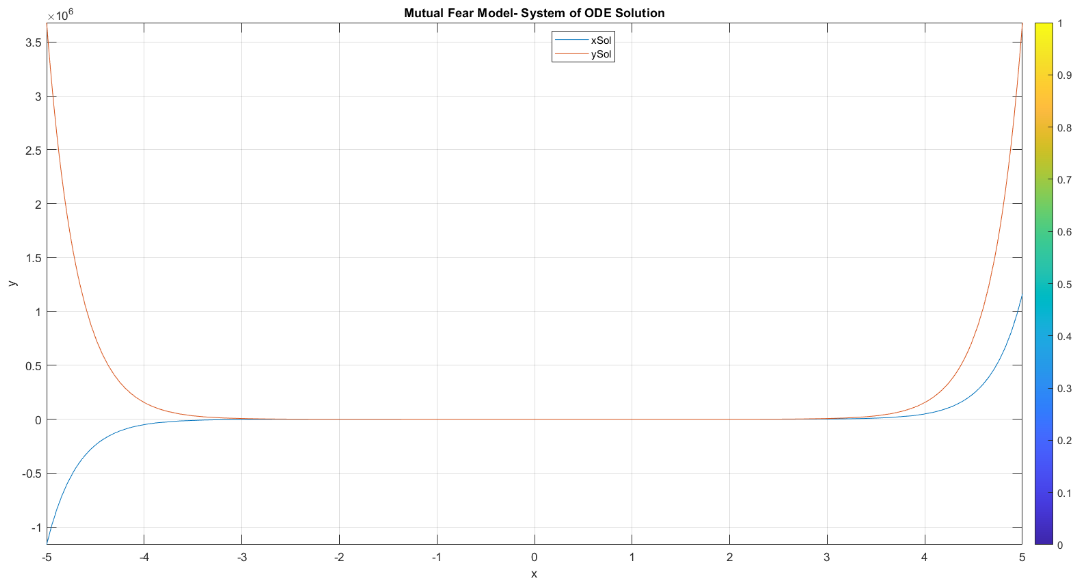


Figure 1.5

Chapter 2

Richardson Model

2.1 Introduction

We now present a refinement of the Mutual Fear Model which we found to produce a "runaway" arms race with unlimited expenditures. To prevent unlimited expenditures, we assume that excessive armament expenditures present a drag on the nation's economy so that the actual level of expenditure reduces the rate of change on the expenditure.

2.1.1 Assumptions

The simplest way to model this is to assume that the rate of change for a nation is directly and negatively proportional to its own expenditure. To model this, we introduce two additional parameters (m and n) and obtain:

$$\frac{dx}{dt} = ay - mx + r$$
$$\frac{dy}{dt} = bx - ny + s.$$

Before proceeding with a mathematical analysis of this model, we introduce a further refinement. This refinement models any underlying grievances of each country toward the other. To model this, we introduce two additional constant terms (r and s) to get:

$$\frac{dx}{dt} = ay - mx + r$$
$$\frac{dy}{dt} = bx - ny + s.$$

A positive value of r and s indicates that there is a grievance of one country toward the other which causes an increase in the rate of arms expenditures. If r and s are negative, then there is an underlying feeling of good will, so there is a decrease in the rate of arms expenditures. This model is called Richardson's Arms Race Model in honour of Lewis F. Richardson, who considered this model in 1939 for the combatants of World War I.

2.2 Solution

The dynamic behavior of this system of differential equations depends on the relative sizes of ab and mn together with the signs of r and s. Although the model is a relatively simple one, it allows us to consider several different long-term outcomes. It's possible that two nations might move simultaneously toward mutual disarmament, with x and y each approaching zero. A vicious cycle of unbounded increases in x and y is another possible scenario. A third eventuality is that the arms expenditures asymptotically approach a stable point (x^*, y^*) regardless of the initial level of arms expenditures. In other cases, the eventual outcome depends on the starting point.

Equilibrium Values

After figuring out the equilibrium values which are given by

$$(x,y) = (\frac{nr+sa}{mn-ab}, \frac{ms+br}{mn-ab})$$

.

This is found out by solving for $(0,0) = (ay - mx + r, bx - ny + s)$
Now finding out the Jacobian of this system of ODEs

$$J_(x,y) = \begin{pmatrix} -m & a \\ b & -n. \end{pmatrix}$$

Now the $\det(J) = mn - ab$ and $tr(J) = -(m+n)$ where $m,n,a,b \in \mathbb{R}^+$ and $r,s \in \mathbb{Z}$

Figure 2.1 shows one possible situation with four different initial levels, each of which leads to a "stable outcome", and the intersection of the nullclines $dx/dt = 0$ and $dy/dt = 0$.

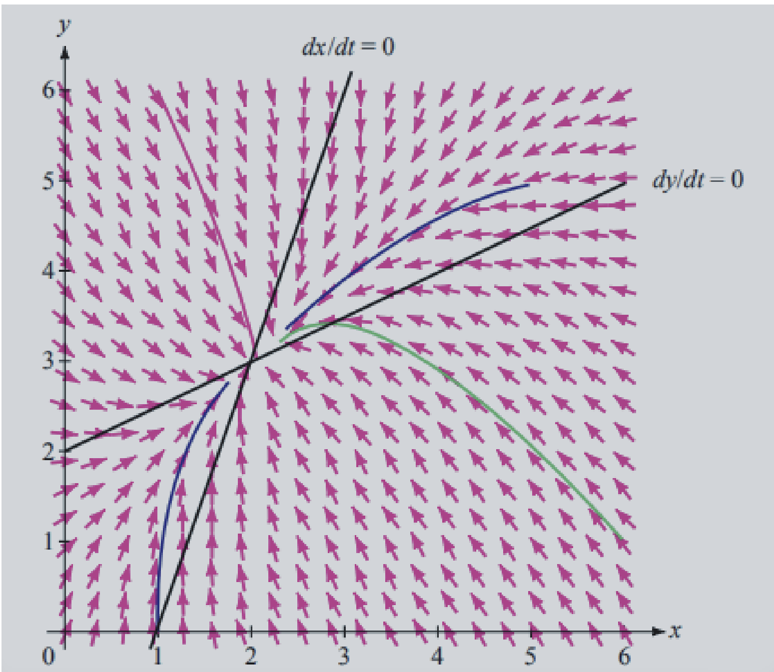
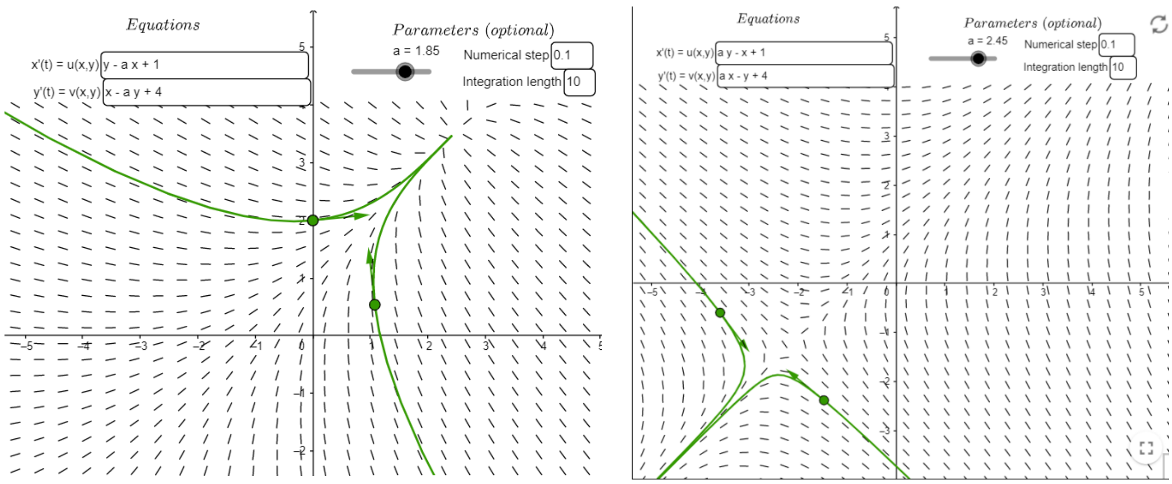


Figure 2.1



(a) $mn > ab$ *(b) $mn < ab$*

Figure 2.2: Mutual Grievances - Phase Potraits

2.2.1 Mutual Grievances

This is when $r, s > 0$, and we also know $tr(J) < 0$, so we just have to analyze the $\det(J)$.

Case 1

When the $det(J) > 0$, then $mn > ab$, and it would imply there is a balance of power between the two nations as both r, s are positive. Hence there would be grievance, but since $mn > ab$, we have a stable node and hence at the equilibrium point (doesn't really matter as the Jacobian is independent of x, y) we have that both the nations have equal power. This lies in the first quadrant as all values are positive.

Case 2

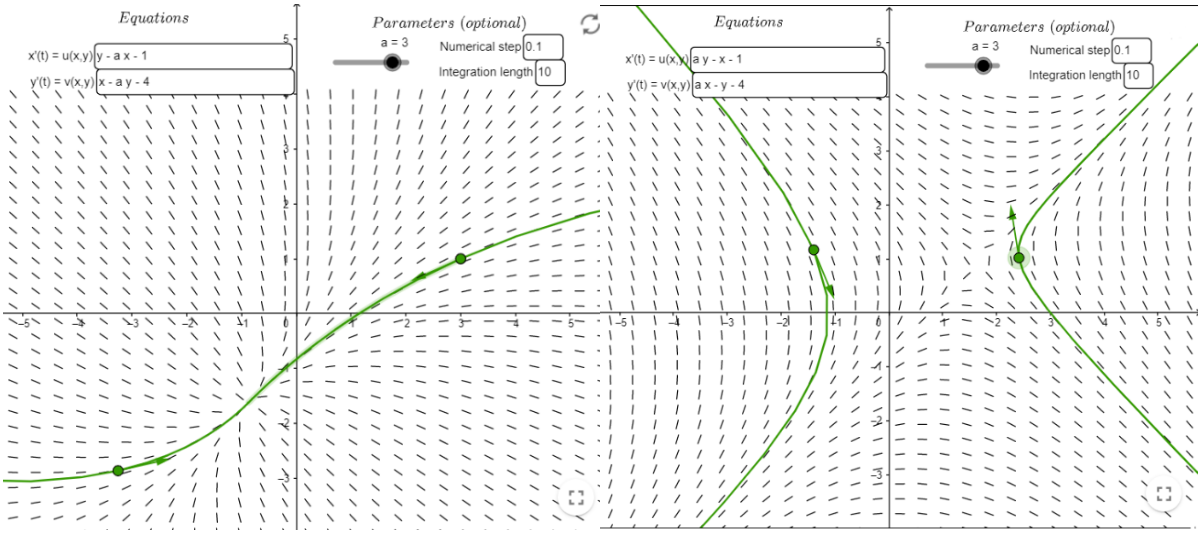
When $\det(J) < 0$, $mn < ab$, which would imply a total arms race as this is an unstable node and hence both the nations increase their total arms collection alongside the fact that there is a huge grievance between the two nations and hence they keep on increasing their arms expenditure. This lies in the third quadrant as the $\det(J)$ is negative.

2.2.2 Feeling of Good Will

When $r, s < 0$, knowing $tr(J) < 0$ we just have to analyze $\det(J)$.

Case 1

When $\det(J) > 0$, $mn > ab$, and this would imply there is a total arms reduction due to the fact that both nations have less grievance with each other as well as a stable node and hence this results in both nations reducing their arms.



(a) $mn > ab$ (b) $mn < ab$

Figure 2.3: Feeling of Good Will - Phase Potraits

Case 2

When $\det(J) < 0$, $mn < ab$. This is a bit of a tricky situation as both nations although have less grievance with each other but due to the sign of the determinant being negative we have an unstable node and hence this is an unstable arms race where nations increase and reduce alternatively and hence a very unstable system of ODEs.

2.3 Critique

The arms need not even be weapons. Colleges have engaged in amenities arms races, often spending millions of dollars on more luxurious dormitories, state-of-the-art athletic facilities, epicurean dining options, and the like to be more competitive in attracting student applications. Biologists have identified the possibility of evolutionary arms races between and within species as an adaptation in one lineage may change the selection pressure on another lineage, giving rise to a counter adaptation. Most generally, the assumptions represented in a Richardson-type model also characterize many competitions in which each side perceives a need to stay ahead of the other in some mutually important measure. *Olinick* (2009)

For some values of the parameters of the Richardson model the predicted value of armament increases without bound. This feature is generally acknowledged as unrealistic. One possibility is that the model is a linear approximation useful within some range. Other possibilities are that eventually, different constraints will come into play that will change the system. Two possible mechanisms for change are the occurrence of war between the nations, or one side abandoning the race. A more complex model will be able to capture a wider range of behavior during the racing period. *Hill* (1992)

There are fiscal constraints that will prevent arms expenditures from increasing without bound. One scenario is that an exploding increase in military outlays levels will eventually result in higher tax burdens, bigger government deficits, and for developing countries dampening investment and reduced savings. Eventually, so many funds arc allocated to the military that the burden placed on others sectors becomes unbearable. In the case of the economic constraint, money devoted to arms will be drained away from other sectors of the economy. Eventually the domestic pressure to devote resources to other sectors of the economy will limit arms expenditures.

Obviously, x, y < 0 also has no physical comparison, and the Richardson Model does not work for every governmental structure.

2.4 Extensions

The basic structure of a multi-nation arms race, in Richardson's terms, much like weather systems, is given as a system of ordinary differential equations as shown below, where x_i is the military spending for nation i, and $\kappa_{i,j}$ has the action-reaction coefficients off the diagonal, and the economic constraints on the diagonal and g_i portrays the hostility terms. *Gleditsch* (2020)

$$\frac{dx_i}{dt} = g_i + \sum_{j=1}^{j=n} k_{ij} x_j \quad \forall \quad i \in \{1,2,3,\ldots,n\}$$

This equation is the crux of the multilateral system of equations. Instead of x and y there is now a vector of countries stored in x_i, where i is an index of all the countries to be included (including the previous y from the bilateral case). κ is an $i \times i$ matrix. The off-diagonal elements collect the action-reaction terms, linking each i to each other i with a coefficient that conveys the reaction of a single country to each other countries' military spending.

Chapter 3

Surveillance Model

3.1 Introduction

Building on the Richardson Model, we now consider an extension where we investigate the extents to which nations go to bring awareness of what their opposition is concealing. In an arms race, one can expect significant surveillance costs associated with determining the armature levels of opposing nations. Roger Bexdek attributes surveillance costs to the National Defense budgets of a nation, which is not necessarily grouped with armature expenditure *Bezdek* (1975). Therefore, we differentiate the two concepts better to understand the economical logistics of an arms race and consider the effects on the ensuing expenditures a nation can make on their military production.

Our objective is to determine the effects of mutual grievance and goodwill thinking between nations when we distinguish between armature and surveillance costs. We will consider how changes in GDP relative to armature expenditure affect our models related to the Nash equilibrium, further addressing the implications of what our model means for the nations at war.

3.2 Formulation of Model

We will begin with the formulation of our model where we established that surveillance expenditure is separate from armature expenditure. With the goal of making our model more realistic we consider the *logistic model*, where the variations are in large part are handled by a limiting factor, that is, the expenditure relative to surveillance. We begin by defining the relevant variables in our model:

$$\begin{aligned}
K_p &= \text{Maximum expenditure of Purple nation on armature} \\
K_g &= \text{Maximum expenditure of Green nation on armature} \\
v_p &= \text{Surveillance expenditure of Purple nation} \\
v_g &= \text{Surveillance expenditure of Green nation} \\
A_p &= K_p + v_p \\
A_g &= K_g + v_g
\end{aligned}$$

Given most environments have a constraint on resources, we want to address the potentials of continuous growth or decay where we experience convergence to an equilibrium point. We enhance our model by A_p such that the armature expenditure x is always less than the defense spending A_p. This adaptation is displayed in Equation 3.1.

$$(1-\frac{x}{A_p}) \tag{3.1}$$

The variable A_p for purple is the adjusted ceiling on defense expenditure that is equal to the maximum armature expenditure K_P plus the surveillance expenditure V_P. This translates our model to the following:

$$\frac{dx}{dt} = a(1-\frac{x}{A_p})y - mx + r \tag{3.2}$$

$$\frac{dy}{dt} = a(1-\frac{y}{A_g})x - ny + r \tag{3.3}$$

where $a, b, m, n > 0$.

While this model is a more realistic demonstration, it is limited in the considerations of more nations being involved, changes in resource availability, new allies or enemies, larger demand for military personnel as well as the indirect factors such as market bubbles and immigration. The issue of involving more nations will be addressed in the next chapter but the other limitations are left for consideration.

3.3 Solution

The objective in this section is to determine the equilibrium points associated with both mutual grievance and the good will effect for our adapted model. Our investigations on the behaviour of our surveillance model is done through phase plane analysis. The purpose is to determine points of stability that nations can employ alongside to determine the Nash Equilibrium of the two nations which will be discussed in a later section.

We begin by isolating our equations,

$$ay - \frac{xy^*}{A_p} - mx^* + r = 0 \tag{3.4}$$

$$bx - \frac{xy^*}{A_g} - ny^* + s = 0 \tag{3.5}$$

from which we determine there are two interpretable equilibrium values,

$$(x_1, y_1) = (\frac{r}{m}, 0) \tag{3.6}$$

$$(x_2, y_2) = (0, \frac{s}{n}). \tag{3.7}$$

In order to determine the stability of our model we consider the Jacobian matrix below. By plugging our equilibrium values into the Jacobian, we can solve for our traces and determinants. This relationship between the trace and determinant will shape our results by confirming or denying stability for any given equilibrium point.

$$J(x,y) = \begin{pmatrix} \frac{-y}{A_p} - m & a - \frac{x}{A_p} \\ b - \frac{y}{A_g} & \frac{-r}{mA_g} - n \end{pmatrix} \tag{3.8}$$

$$J(\frac{r}{m},0) = \begin{pmatrix} -m & a - \frac{r}{mA_p} \\ b & \frac{-r}{mA_g} - n \end{pmatrix} \tag{3.9}$$

$$J(0,\frac{s}{n}) = \begin{pmatrix} \frac{-s}{nA_p} - m & a \\ b - \frac{s}{nA_g} & -n \end{pmatrix} \tag{3.10}$$

Using the above results, we consider the two scenarios of mutual grievance and good will in the following sections.

3.3.1 Mutual Grievances

We first perform phase plane analysis where $r,s > 0$ and the positive natures of these parameters means that there are mutual grievances, that is, each nation is increasing armature expenditure. Solving for the trace and determinant when our equilibrium points are $(\frac{r}{m},0)$, we get the following:

$$tr(J) = -(m + \frac{r}{mA_g} + n) \tag{3.11}$$

$$det(J) = (mn - ab) + r(\frac{1}{A_g} + \frac{b}{mA_p}) \tag{3.12}$$

Given all our parameters are positive values as stated in previous chapters, we conclude $tr(J) < 0$. However, there are a few situations when it comes to the $det(J)$, namely,

$$mn - ab > 0 \text{ implies } det(J) > 0$$

$$mn - ab < 0 \text{ where } |mn - ab| < r(\tfrac{1}{A_g} + \tfrac{b}{mA_p}) \text{ implies } det(J) > 0$$

$$mn - ab < 0 \text{ where } |mn - ab| > r(\tfrac{1}{A_g} + \tfrac{b}{mA_p}) \text{ implies } det(J) < 0$$

The explicit substitutions and detailed calculations of the above can be found in (**A.n**). The proceeding results are derived in an identical manner.

When scenarios (i) and (ii) hold, we have negative real eigenvalues and thus stability for equilibrium $(\frac{r}{m}, 0)$. In scenario (iii) our equilibrium is unstable. Repeating this process for $(0, \frac{s}{n})$,

$$tr(J) = -(n + \frac{s}{nA_p} + m) \tag{3.13}$$

$$det(J) = (mn - ab) + s(\frac{1}{A_p} + \frac{a}{nA_g}) \tag{3.14}$$

Given all our parameters are positive values, $tr(J) < 0$. There are again a few situations when it comes to the $det(J)$ however,

- $mn - ab > 0$ implies $det(J) > 0$
- $mn - ab < 0$ where $|mn - ab| < s(\frac{1}{A_p} + \frac{a}{nA_g})$ implies $det(J) > 0$
- $mn - ab < 0$ where $|mn - ab| > s(\frac{1}{A_p} + \frac{a}{nA_g})$ implies $det(J) < 0$

We see from scenarios (i) and (ii) above, we have negative real eigenvalues and stability for the equilibrium $(0, \frac{s}{n})$. In scenario (iii) however, our equilibrium is unstable.

3.3.2 Good Will Effect

In a similar analysis to the Mutual Grievances, we consider $r, s < 0$. The negativity of which suggests that there is a good will effect where each nation is decreasing armature expenditure.

Using $(\frac{r}{m}, 0)$ and given all our parameters are positive values ($tr(J) < 0$), we investigate the $det(J)$ and the following cases:

- $mn - ab < 0$ implies $det(J) < 0$
- $mn - ab > 0$ where $mn - ab < |r(\frac{1}{A_g} + \frac{b}{mA_p}|)$ implies $det(J) < 0$
- $mn - ab > 0$ where $mn - ab > |r(\frac{1}{A_g} + \frac{b}{mA_p}|)$ implies $det(J) > 0$

Thus, when scenario (iii) holds, we have negative real eigenvalues and a stability for the equilibrium value $(\frac{r}{m}, 0)$. In scenarios (i) and (ii), our equilibrium is unstable here.

For the equilibrium point $(0, \frac{s}{n})$, we found the following:

- $mn - ab < 0$ implies $det(J) < 0$
- $mn - ab > 0$ where $mn - ab < |s(\frac{1}{A_p} + \frac{a}{nA_g})|$ implies $det(J) < 0$
- $mn - ab > 0$ where $mn - ab > |s(\frac{1}{A_p} + \frac{a}{nA_g})|$ implies $det(J) > 0$

Where (iii) shows negative real eigenvalues and a stability for the equilibrium value. In (i) and (ii), our equilibrium is stable.

3.4 Interpretation of results

We will now explore the various implications of our results, this begins with the discussion on the reoccurring relationship between mn and ab.

3.4.1 GDP and Defense Expenditure

The term $mn - ab$ appears consistently throughout our results, this is a result of its relationship with the rate of change in expenditure for both nations and the response to the oppositions armature levels. We know a, b is directly proportional to the opposing nation, and m, n is negatively proportional to the nations own expenditure. More simply, mn is the rate of change in the expenditure of both nations and ab is the rate of change for a nations production relative to the other. More relevantly, this is a comparison of the growth of GDP to the production of arms. If GDP is growing faster than armament, we have a positive relationship, conversely, if armament grows faster than GDP, we have a negative relationship. When both grow proportionally the cases become easier to interpret and has complete dependence on the sign of r and s alone.

Our model predicts when mutual grievances are present and GDP growth is larger than armature expenditure. We can expect stability to develop given the Green nation doesn't spend on armature and the Purple nation does ($\frac{r}{m} > 0$).

Alternatively, when goodwill is present and GDP growth is larger than armature expenditure, we can expect stability to develop given the Purple nation doesn't spend on armature and the Green nation does $\frac{s}{n} < 0$.

3.4.2 Effect of Surveillance Expenditure

Recall the relationships $A_p = K_p + v_p$ and $A_g = K_g + v_g$ where, each nations defense expenditure includes their individual surveillance costs. Using our previous analysis this tells us,

1. The higher surveillance spending, the larger our A_p or A_g values are. Given the situation of mutual grievances, this implies $s(\frac{1}{A_p} + \frac{a}{nA_g})$ and $r(\frac{1}{A_g} + \frac{b}{mA_p})$ will be positive but not substantially large. Conversely, goodwill thinking implies $s(\frac{1}{A_p} + \frac{a}{nA_g})$ and $r(\frac{1}{A_g} + \frac{b}{mA_p})$ will be much smaller and negative, making stability easier to achieve in both cases given constant GDP and armature expenditure.

2. Similarly, the lower the surveillance spending, the smaller our A_p or A_g values. Given the situation of mutual grievances, this implies $s(\frac{1}{A_p} + \frac{a}{nA_g})$ and $r(\frac{1}{A_g} + \frac{b}{mA_p})$ will be positive and larger than before. Conversely, goodwill thinking implies $s(\frac{1}{A_p} + \frac{a}{nA_g})$ and $r(\frac{1}{A_g} + \frac{b}{mA_p})$ will be substantially smaller and negative making stability harder to achieve in both cases as a result of the more extreme changes.

In both cases, higher surveillance spending is a benefit when it comes to stability. We can theorize that the more a nation spends on surveillance given the previously described conditions for stability, the more consistent that nations relationship will be to the opposition. Implying fewer erratic responses on either side. This leads to easier predictability on a nations responses to another as a result of the feedback loop of stable responses. In the next portion we consider what the opposing nations will likely choose to optimize their defense expenditure.

3.4.3 Nash Equilibrium of Defense Spending

From previous sections we know which equilibrium a nation will converge to based on the specified condition of how they are feeling, that is, mutual grievances or good will. We now consider what the nations will choose in a strategic game environment. We first establish a set of assumptions:

1. Each nation is unboundedly rational.
2. Nations have incomplete information regarding their opponent, that is, each nation knows their opponent's payoff but not the actions that are taken against them.
3. Each nation chooses the mutual grievances or good will feeling based on the signals of the opposing nation.
4. If nations have opposing feelings, they continue to surveil without taking action.

Nash Equilibrium

The Nash Equilibrium describes an outcome in which no player wishes to change their strategic choice given the strategy of their opponent. This is a result of no such deviation from the equilibrium that is profitable. As described by Charles Holt and Alvin Roth, the intent is to find the optimal payoff alone, unresponsive to external conditions outside of the game environment *Holt and Roth* (2004).

Considering our results in terms of payoffs, we have the following array of relationships between Mutual Grievances (MG) and Good Will (GW):

		Green	
		MG	GW
Purple	MG	(r/m,0)	$(0,0)$
	GW	$(0,0)$	(0,s/n)

Table 3.1: Payoff Matrix for Purple and Green Nation

Using a best-response approach and strictly dominated analysis, where we determine the dominant strategy of two choices based on the better payoff, it is easy to determine that (MG, MG) is the Nash Equilibrium. We now consider *Pareto Dominance* to solve this. In Pareto dominance, the refinement of the goal is to select the payoff that makes it so any one individual is not better off without making at least one other individual worse off. This concept will be explained more in-depth when we analyze our combined models. In this method, the Nash Equilibrium still remains the same as the previous scenarios. Selecting (GW, GW) would result in negative expenditure and conversely (MG, MG) results in positive expenditure. While you may notice more expenditure does not equate to more returns, it does imply more defense, which has the alternative benefit of becoming more of a military power compared to the other nation. The implications include less concerns of arms race expenditure with other countries and appearing more authoritative.

If both nations choose the optimal strategy of sending MG signals, Purple nation will increase armature and surveillance spending which also suggests less saving on defense expenditure, thus, less future funding. This coincides with the potential to switch to a GW mentality if lack of funds becomes a constraint. This implies that the more that's spent on surveillance and/or armature, which adds to defense spending, the longer the nation can maintain the Nash Equilibrium. In the case Purple nation cannot afford the Nash Equilibrium, they will switch to the payoff (GW, GW). This way they are not increasing or decreasing defense spending while the opposing nation decreases their defense spending. In an arms race, this is an ideal backup plan. Purple nation can save funds, however Green nation is decreasing expenditure which also supports growth of funds.

The the Nash Equilibrium and the fallback payoff have the ability to cause a negative feedback loop. Switching from pure strategies based on affordability and signals, nations receive signals regarding an opposing nation's strategy through surveillance. The more spent on surveillance, the stronger their beliefs are regarding the likelihood of the opposing nation's choices. However, this does have a consequence of a nation having fewer funds to increase armature spending if needed, this is a danger to the Purple nation but not the Green nation. From a mathematical and economic standpoint, the Green nation has a better opportunity for armature success assuming no nation is actually going to war and with respect to risk neutrality. We interpret this as Purple nation being more of a risk taker than Green nation.

3.5 Critique of Model

The model in this chapter has provided substantial implications and understanding of how the use of a logistic model in an arms race enhances our results. We were able to determine the effect of mutual grievances and good will thinking on our model when we provide a distinction between armature and surveillance costs. We then considered how changes in GDP relative to armature expenditure effects the model for either nation. Finally, we discussed our models relationship to the Nash Equilibrium where we mitigated the strategic impact of diverting from the equilibrium. The model sufficiently addressed our objectives and provided a continuation of results. While we are limited by the assumptions of our Nash Equilibrium, there are factors we have not considered such as resource inflation in the market, or the effect of defense contractors. One thing we can improve for this model is the consideration of a third nation. We will do this in the next chapter.

Chapter 4

Three Nation Model

4.1 Introduction

This extension involves the introduction of a third nation to the system of difference equations. This reflects the extension of the Richardson model for 3 nations but with an additional factor of a base level wherein, without any interactions between nations, the production reverts to a base level over time. The longer the war between these three nations goes, the further each nation goes to display dominance in the levels of destruction each new weapon can cause. A generalization at the end will be given on how to extend it for an n nation model and further suggestions of various other models.

4.1.1 Questions and Assumptions

- Three-nation arms race models are fundamentally unstable because any two nations can form an alliance against the third. In this case how do we modify our original model?

- Our assumptions are based on the fact that all of our thinking is dichotomous, in other words, a nation either has allies or enemys.

- We also assume that each nation has excellent intelligence on each other, and sets arm expenditures on the basis of its knowledge on the expenditure of the other. The key to this predictive policy is our assumption that all nations believe that a war between equals is detrimental to all. This implies that a nation will not attack unless they have some definitive margin over the other. In particular, we assume that a nation is threatened if the enemy has the margin to attack, and feels safe if they do not. As such, a nation will increase its production expenditures to the extent where it feels threatened, and likewise will decrease the expenditure until it feels safe.

- This model raises a lot of questions - How many of the arms races in history are three sided rather than two? How many involve nations with the sole intent of conquest?

- This model is completely linear, this was done for analytic purposes as the simplicity of the conclusions is lost when the model is generalized and multiple equilibrium points exist. For instance, among three countries, the equations for each pair of nations may be stable, but the triplet is unstable.

- There is no psychological influence and we assume that all nations act on their own best interest.

- Economic factors have not been taken into consideration such as the earlier case in the surveillance model

4.2 Formulation

Consider the Richardson's model when there are three nations to the conflict

$\frac{dx}{dt} = m_x(\bar{x} - x) + r_{xy}y + r_{xz}z$

$\frac{dy}{dt} = m_y(\bar{y} - y) + r_{yx}x + r_{yz}z$

$\frac{dz}{dt} = m_z(\bar{z} - z) + r_{zy}y + r_{xz}x$

Denoted by x, y and z, the measure of nation X, Y and Z's armament towards the other respecitively. Assume that without interaction, this reverts over time to a base level denoted by $\bar{x}, \bar{y}, \bar{z}$. Now let m_x, m_y, m_z denote the speed of adjustment to the base level. This base level is, however, not the long-term equilibrium value as each nation will interact with each other in a way. Let r_{ij} denote nation i's reaction coefficient to nation j's state where $i, j \in \{x, y, z\}$ and $i \neq j$.

4.2.1 Assumptions

- Now we assume that $\forall r_{ij}$ $i, j \in \{x, y, z\}$ and $i \neq j$ there are positive r_{ij}. This is because for a particular nation they induce additional military procurement due to the other nations government in the case when other nations have high levels of armament.

- Let us simplify our model even further by assuming $r_{ij} = r_{ji} = r$ and $m_x = m_y = m_z = m$ and assume that $m > r > 0$. This is due to the fact that the analysis of a 3D ODE system is pretty hard to do and simplifying this further helps us interpret the model even better. We will relax the conditions when we further suggest other improvements.

- We assume that $m > r > 0$ is due to the fact that if we allow negative values of r and m, it leads to a more complicated situation which results in a chaotic system. $m > r$ means that the rate at which the nation returns back to the base level is always going to be greater than reaction coefficients between any two nations.

4.3 Solution

Now a 3 dimensional system is pretty hard to analyse but there are theorems which can help us such as the Routh-Hurwitz theorem and the Lyapunov Function method. We will go over using the Routh- Hurwitz theorem first for our case. *Math24* (2019)

Routh-Hurwitz theorem

Suppose we are given an nth order homogeneous system of differential equations with constant coefficients:

$$\mathbf{X}'(t) = A\mathbf{X}(t), \mathbf{X}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \vdots & a_{nn} \end{bmatrix}$$

where $\mathbf{X}(t)$ is an n-dimensional vector containing the unknown functions, and A is a square matrix of size $n \times n$

A nonlinear autonomous system can be reduced to the linear system by performing a linearization around an equilibrium point. Then without loss of generality, we may assume that the equilibrium point is at the origin and it is always possible to reach by choosing a suitable coordinate system. The stability or instability of the equilibrium state is determined by the signs of the real parts of the eigenvalues of A. To find the eigenvalues λ, it is necessary to solve the auxiliary equation

$$\det(A - \lambda I) = 0$$

which is reduced to an algebraic equation of the nth degree,

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0.$$

In such a situation, methods allowing us to determine whether all roots have negative real parts and establish the stability of the system without solving the auxiliary equation itself, are of great importance. One of these methods is the Routh-Hurwitz criterion, which contains the necessary and sufficient conditions for the stability of the system. Consider again the auxiliary equation that describes the dynamic system.

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

Note, the necessary condition for the stability is satisfied for all of the coefficients where $a_i > 0$. Therefore, we assume that the coefficient $a_0 > 0$, and We have the so-called "Hurwitz matrix".

The main diagonal of the matrix contains elements $a_1, a_2, \ldots, a_n$. The first column contains numbers with odd indices $a_1, a_3, a_5, \ldots$ in each row, and the index of each following number (counting from left to right) is 1 less than the index of its predecessor. All other coefficients a_i with indices greater than n or less than 0 are replaced by zeros.

The result is a matrix of kind:

$$\begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \vdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & a_n \end{bmatrix}.$$

The principal diagonal minors Δ_i of the Hurwitz matrix are given by the formulas:

$$\Delta_1 = a_1, \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & \vdots & 0 \\ a_3 & a_2 & a_1 & \vdots & 0 \\ a_5 & a_4 & a_3 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & a_n \end{vmatrix}.$$

We now formulate the Routh-Hurwitz stability criterion; the roots of the auxiliary equation have negative real parts if and only if all the principal diagonal minors of the Hurwitz matrix are positive, provided that $a_0 > 0 : \Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0$. As $\Delta_n = a_n \Delta_{n-1},$, the last inequality can be written as $a_n > 0$.

In our case we have a 3rd order system, the stability criterion is defined by the inequalities:

$$a_0 > 0, \Delta_1 = a_1 > 0, \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0, \Delta_3 = a_3 > 0$$

or

$$a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0, a_1 a_2 - a_0 a_3 > 0$$

If all the $n-1$ principal minors in the Hurwitz matrix are positive and the nth order minor is zero ($\Delta_n = 0,$), the system is at the boundary of stability. As $\Delta_n = a_n \Delta_{n-1},$, there are now two options:

- The coefficient $a_n = 0$. This corresponds to the case when one of the roots of the auxiliary equation is zero. The system is on the boundary of the aperiodic stability.

- The determinant $\Delta_{n-1} = 0$. In this case, there are two complex conjugate imaginary roots. The system is on the boundary of the oscillatory stability.

Cases

Let us now calculate the Jacobian of this system of ODEs as:

$$J_(x,y,z) = \begin{pmatrix} -m_x & r_{xy} & r_{xz} \\ r_{yx} & -m_y & r_{yz} \\ r_{zx} & r_{zy} & -m_z \end{pmatrix}$$

.

Then, finding the eigenvalues of this system is difficult but leads to the characteristic equation

$$\lambda^3 + (m_x + m_y + m_z)\,\lambda^2 + (m_x m_y + m_x m_z + m_y m_z - r_{xy}r_{yx} - r_{zx}r_{xz} - r_{yz}r_{zy})\,\lambda + \gamma = 0$$

where

$$\gamma = m_x m_y m_z - m_z r_{xy} r_{yx} - m_y r_{xz} r_{zx} - m_z r_{zx} r_{xz} - r_{xz} r_{yx} r_{zy}.$$

Clearly, the Wolfram Alpha solution shows that this system of ODE has one real root and 2 complex roots.

Let us apply the Routh-Hurwitz theorem to this equation now. We have our characteristic equation,

$$\lambda^3 + (m_x + m_y + m_z)\,\lambda^2 + (m_x m_y + m_x m_z + m_y m_z - r_{xy}r_{yx} - r_{zx}r_{xz} - r_{yz}r_{zy})\,\lambda + \gamma = 0$$

where,

$$\gamma = m_x m_y m_z - m_z r_{xy} r_{yx} - m_y r_{xz} r_{zx} - m_z r_{zx} r_{xz} - r_{xz} r_{yx} r_{zy}$$

that gives us,

- $a_0 = 1$
- $a_1 = (m_x + m_y + m_z)$
- $a_2 = (m_x m_y + m_x m_z + m_y m_z - r_{xy}r_{yx} - r_{zx}r_{xz} - r_{yz}r_{zy})$
- $a_3 = \gamma$
  $a_4 = 0$

which we put into the Hurwitz matrix, $\begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{bmatrix}$ and find all its principal minors as

- $\Delta_1 = (m_x + m_y + m_z) > 0$
- $\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 \cdot a_2 - a_3 \cdot a_0$
- $a_3 \cdot a_0 = \gamma$
- $a_1 \cdot a_2 = (m_x + m_y + m_z) \cdot (m_x m_y + m_x m_z + m_y m_z - r_{xy}r_{yx} - r_{zx}r_{xz} - r_{yz}r_{zy})$
- $\Delta_3 = \Delta_2 \cdot a_3.$

Case 1

The signs of the real parts can be both positive and negative depending on the parameters of the system - in particular on the relative sizes of the m and r.

Now, applying our assumptions $r_{ij} = r_{ji} = r$ and $m_x = m_y = m_z = m$ and $m > r > 0$, we see

- our principal minors become $\Delta_1 = (3m) > 0$ and

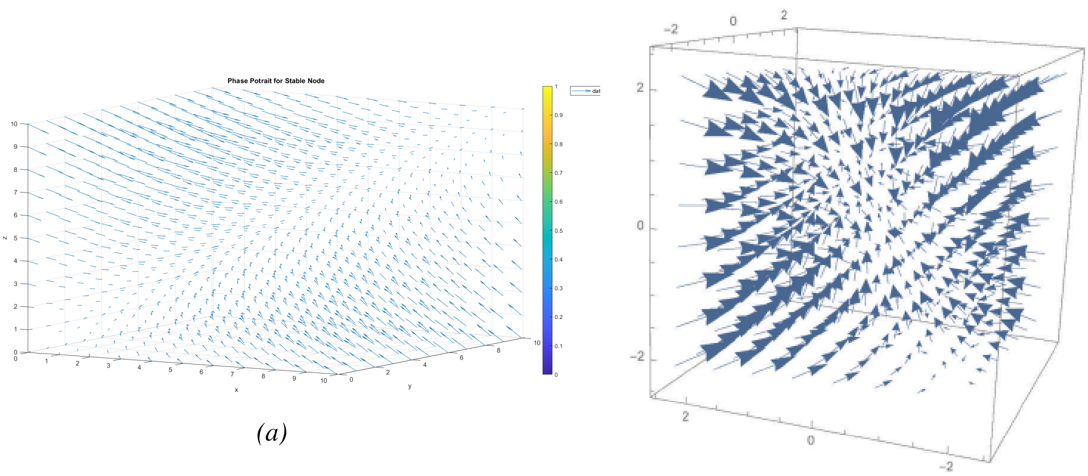
  $$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 \cdot a_2 - a_3 \cdot a_0.$$

- To find $a_3 \cdot a_0 = \gamma = m^3 - 3mr^2 - r^3$ and $a_1 \cdot a_2 = (3m) \cdot \left(3m^2 - 3r^2\right)$ since $m > r$ we get, $a1 \cdot a2 > 0$

- Therefore $\Delta_2 = 9m^3 - 9mr^2 - m^3 + 3mr^2 + r^3 = 8m^3 - 6mr^2 + r^3 > 0$ as $m > r$ and $\implies m^3 > r^3$, hence $\Delta_2 > 0$.

- Finally, we find $\Delta_3 = \Delta_2 \cdot a_3 = (8m^3 - 6mr^2 + r^3)(m^3 - 3mr^2 - r^3) > 0$ by a similar analysis as above.

Therefore this is clearly a stable node by the theorem. Now the Matlab phase portrait yields the case when $m = \frac{1}{10}$ and $r = \frac{1}{20}$.



(a)

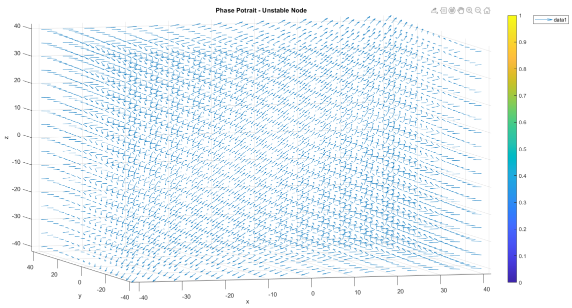
(b)

Figure 4.1: Stable Node

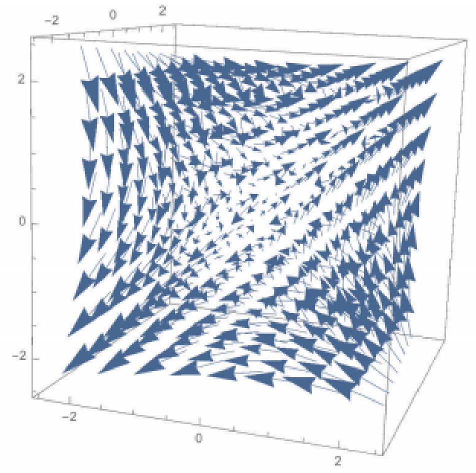
Case 2

Now let us consider the case when $0 < m < r$ then this yields an unstable node as the principal minor $\Delta_2 < 0$ and $\Delta_3 < 0$.

The MATLAB phase portrait yields the case when $m = \frac{1}{20}$ and $r = \frac{1}{10}$



(a)



(b)

Figure 4.2: Instability in the Positive Orthant

Case 3

Now consider the case when one of the nations acts as the peace-maker and hence reduces the escalation levels of other parties. Let us assume nation z is the one acting as the peace maker. Now this state z can be interpreted as an involvement level which depends positively on the level of aggression by other nations.

Now our new assumptions would be that $r_{zy}, r_{zx} > 0$,$r_{xz}, r_{yz} < 0$, $r_{xy}, ryx >$ 0 and $m_x = m_y = m_z = m$ assuming that $|r_{ij}| = r > m > 0$.

- Then our principal minors become $\Delta_1 = (3m) > 0$ and

  $$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 \cdot a_2 - a_3 \cdot a_0$$

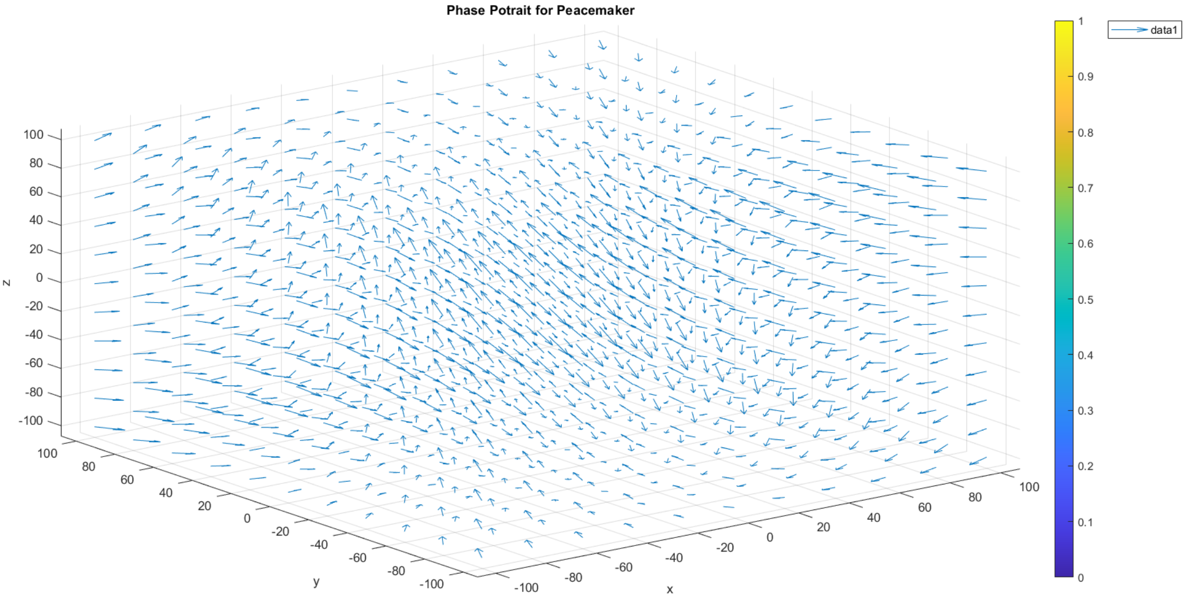
- Now to find $a_3 \cdot a_0 = \gamma = m^3 - mr^2 + r^3$ and $a_1 \cdot a_2 = (3m) \cdot \left(3m^2 + r^2\right)$, since $m > r$ we get $a1 \cdot a2 > 0$

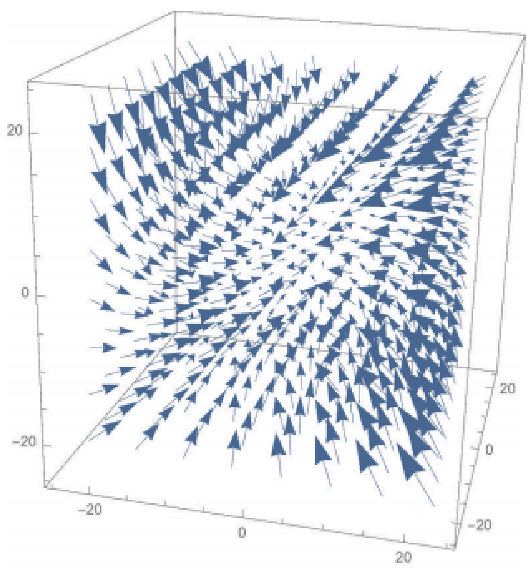
- Therefore $\Delta_2 = 9m^3 + 3mr^2 - (m^3 - mr^2 + r^3) = 8m^3 + 4mr^2 - r^3 > 0$ as $m > r$ and $\implies m^3 > r^3$, hence $\Delta_2 > 0$.

- Lastly, to find $\Delta_3 = \Delta_2 \cdot a_3 = (8m^3 + 4mr^2 - r^3)(m^3 - mr^2 + r^3) > 0$ as $\gamma > 0$.

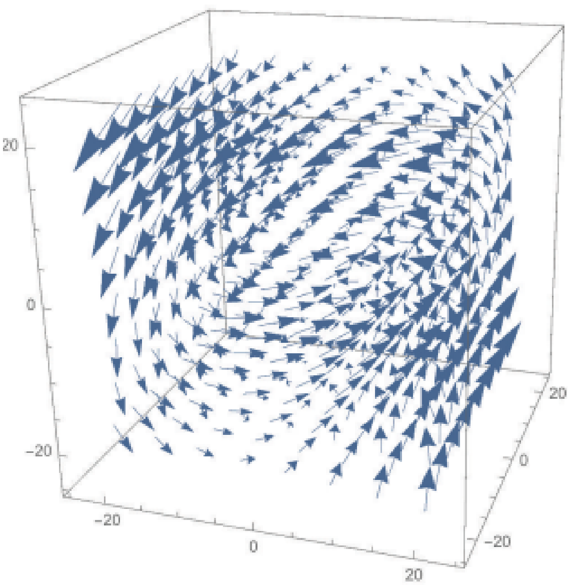
Now the MATLAB phase portrait yields the case when $m = \frac{1}{20}$ and $|r| = \frac{1}{10}$ which yields a centre and hence a stable node.

Stable nodes also display more complex dynamic behaviour than in the two-party case. For example, in the special case where $m_x = m_y = m_z$ and $r_{zx} = r_{zy} = 0.2$ while $r_{xy} = 0.3, r_{yx} = 0.1$ as well as $r_{xz} = r_{yz} = -0.2$, the numerical eigenvalues (with a precision of $10^{-3}) are \lambda_1 = -0.688, \lambda_2 = -0.406 + 0.276i$ and $\lambda_3 = -0.406 - 0.276i$. The resulting asymmetric trajectories are illustrated in the vector phase portrait as shown beside the earlier case.





(a) Stable node with asymmetric patterns



(b) A center

Figure 4.3: Dynamic Patterns of Stable Nodes

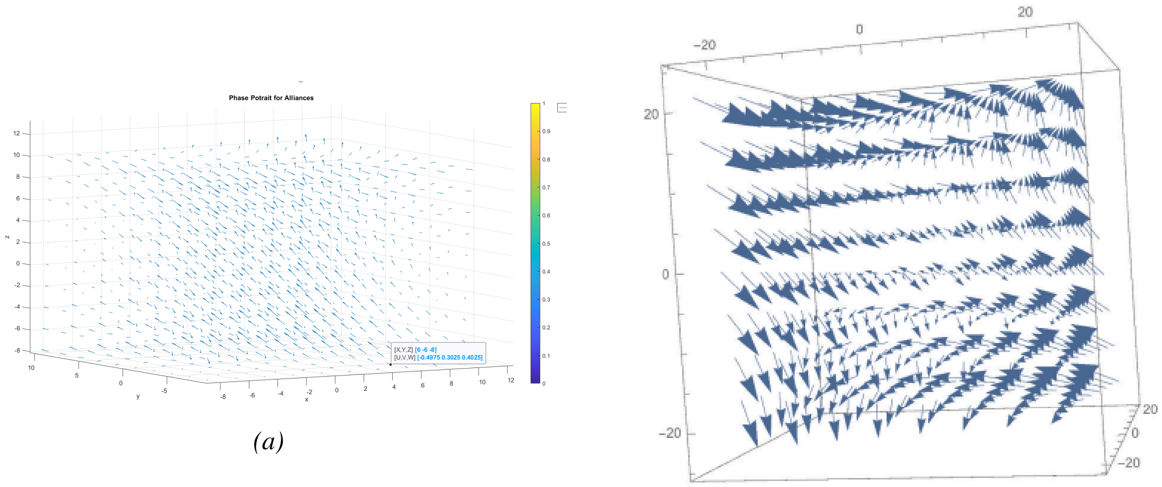
Case 4

In this case we we have that in an alliance, higher armament by one of the parties will lead to a reduction of effort by the other, due to the familiar incentive to rely on the contributions of others. This would imply that if x and y are allies in an otherwise standard conflict with z, we would assume that $r_{xy}, r_{yx} < 0$ while all other coefficients remain positive.

Now our new assumptions would be that $r_{zy}, r_{zx} > 0$, $r_{xz}, r_{yz} > 0$, $r_{xy}, ryx < 0$, $m_x = m_y = m_z = m$ and $|r_{ij}| = r > m > 0$. Resulting in:

- Our principal minors becoming $\Delta_1 = (3m) > 0$ and
  $$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 \cdot a_2 - a_3 \cdot a_0$$
- Then find $a_3 \cdot a_0 = \gamma = m^3 - 3mr^2 + r^3$ and $a_1 \cdot a_2 = (3m) \cdot \left(3m^2 - 3r^2\right)$, since $r > m$ we get $a1 \cdot a2 < 0$
- Therefore $\Delta_2 = 9m^3 - 9mr^2 - m^3 + 3mr^2 + r^3 = 8m^3 - 6mr^2 - r^3 > 0$ as $r > m$ and $\implies r^3 > m^3$ and hence, $\Delta_2 < 0$.
- Lastly, finding $\Delta_3 = \Delta_2 \cdot a_3 = (8m^3 - 6mr^2 + r^3)(m^3 - 3mr^2 - r^3) < 0$ by a similar analysis to the above.

Now the MATLAB phase portrait yields the case for $m = \frac{1}{20}$ and $|r| = \frac{1}{10}$, meaning an unstable vortex and hence an unstable node with three imaginary eigenvalues - two of which have negative real parts.



(a) *(b)*

Figure 4.4: An unstable vortex in the presence of alliance

4.4 Interpretation

Now clearly we can consider different situations and interpret the complex model in different ways.

- $m > r > 0$ we got the fact that it is a stable node which implies that all the nations are in equilibrium with each other (i.e there is no increase in armament and hence all nations revert back to their base level armament)

- $r > m > 0$ now this is clearly an unstable node as mentioned above and in this case we have an explosion of aggression between all three nations resulting in each nation increasing their armament level.

- Now consider the case when one of the nations acts as the peace-maker and hence reduces the escalation levels of the other two parties. In this case, we see it results in the emergence of a centre with cyclical trajectories and a periodic motion of the armament levels. This however implies that all three nations increase and decrease simultaneously depending on how well the peace-maker acts to reduce the armament levels.*Beckmann Klaus* (2016)

- Regarding the same scenario as above, we can also expect stable nodes to be more dynamic and hence resulting in asymmetric trajectories.

- In an alliance we would expect that there is a reduction of armament between two nations but not a complete reduction if the other nation is aggressive. This would imply an increase in the arms race between these two systems as one, excluding the other nation. There is also the case when one of the nations in the alliance will lead to a reduction of effort in arms as they believe that the other nation can increase their arms to cover up the loss of their own. In our analysis, this lead to an unstable vortex which would imply that if the alliance is weak when both parties don't equally contribute, there is increase in armaments between the alliance and the isolated nation. Further, if there is a strong alliance then there is a decrease dependent on the effect the alliance has on the isolated nation.

4.5 Critique

There are several limitations in the model that we have presented due to the fact that the assumptions that have been made.

- Restriction on parameters, such as assuming $m_x = m_y = m_z = m$ and similarly for r_{ij}.

- Assuming the base level is a fixed constant.

- The lack of psychological effects between nations that would simulate a real world situation where sympathy and enmity can transpire into permanent friendship or hatred.

- The assumption that all coefficients are mostly non-negative, removing this restriction yields more scenarios which can be taken into consideration.

- The interaction effect is taken to depend on the escalation of aggression experienced by a nation rather than its level.

- The higher the dimensions of an n-nation model, the harder it becomes to understand the phase portraits as it results in a chaotic system.

- These equations are merely a description of what nations would do if they did not stop to think and in a realistic situation most leaders of nations would discuss with each other without putting the nations citizens life at risk and escalating the tensions between themselves and other nations.
- The model characterizes nations in modern terms as defensively oriented. There is no provision in the model for planned conquest.

4.6 N - Model extension

Similarly we can extend this model into an n-nation model by defining the same variables as above but an extended version and hence the model then looks like this

$$\dot{x_1} = m_{x_1}(\bar{x_1} - x_1) + \sum_{k=1}^{n} r_{x_1 \cdot x_k} x_k$$
$$\dot{x_2} = m_{x_2}(\bar{x_2} - x_2) + \sum_{k=1}^{n} r_{x_2 \cdot x_k} x_k$$
$$\cdots$$
$$\dot{x_n} = m_{x_n}(\bar{x_n} - x_n) + \sum_{k=1}^{n} r_{x_n \cdot x_k} x_k$$

where $x_k \cdot x_k = 0 \; \forall k \in \{1, 2 \cdots n\}$.

Similarly, we can use the Routh-Hurwitz theorem for this n-nation system and proceed to analyze and get further interpretations on the relaxed conditions.

The figure below shows a 10 dimensional system in the original case of the Multi-arm Richardson Model. *Gleditsch* (2020)

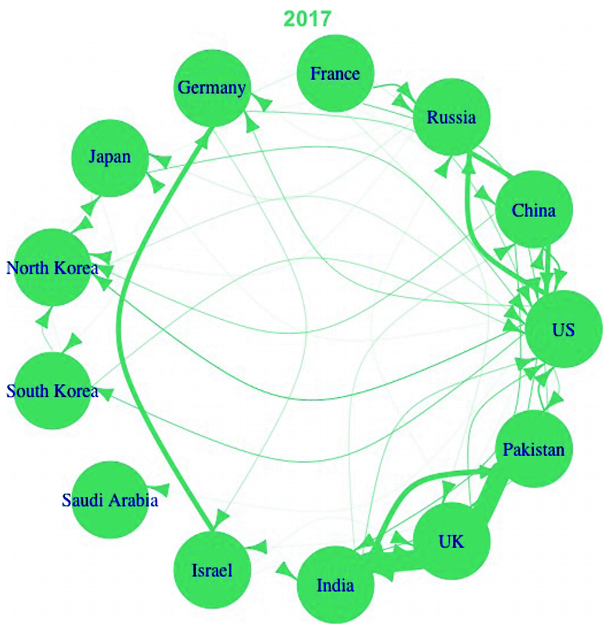


Figure 4.5: Richardsons 10 Dimensional system model

Chapter 5

Combined Model

5.1 Introduction

In this chapter, we bring our models to the pinnacles of their impact in pursuit of realizing what the human race is capable of as a collective military empire. We will tie all of the strings we have attached to the fear-mongering arms expenditure approach in a combined model and establish a set of conditions that predict potential outcomes. Our motivation at this stage lies in the hypothetical circumstances to which the human race would be inclined to concentrate their efforts on presenting the largest military arsenal, such as those of an intergalactic war that would pit the human race against aliens. Given the reliance of our discussions on a nation's armature rates as a result of fear-mongering from any opposition and the various economic deterrence's, the transformation of international initiatives begins at the global market.

5.1.1 Coase Theorem

Amid an arms race, competition is the greatest common denominator in the perception of which nation is at an advantage. In a global market, this competition is contingent on the volume of resources a nation utilizes in production. The competing demands for the same elements mean the value of resources skyrockets. The market dictates prices that decide the justifiable expenditures on production levels that alter how competitive a nation can be. When nations come to the table that translates their competitive spirits to embrace the most effective production rates collectively, we have a seemingly competitive market seeking the most efficient solution on economic allocations. Since the global market is fixated on the single objective of maximal military power the largest military arsenal we can assume no transaction costs on resources and establish a simple process that optimizes a set of inputs and outputs for efficiency. Ronald Coase theorized the above-market would naturally configure the most efficient solution *Coase* (1960).

5.1.2 Pareto Efficiency

With an understanding that all nations must learn to share the pie, Vilfredo Pareto theorized that the global market was seeking a specific "Pareto efficiency", where no alterations on rates would increase production for anyone nation without a loss to another *Furlan* (1908). At the frontier of military production, the limitations of higher overall output would inevitably become a lack of resources. As such, any intentions for the greatest global armature rate mean the resource limitations must reach stability that minimizes its negative impact on each nation's armature rate.

5.1.3 Resource Harvesting

The resource crisis on Earth has been a point of contention for the global leaders for much of the 21st century; various studies into the effectiveness and logistics of accessing new reserves have solicited all manners of debate *Oberle and Clement* (2020) Ref (2009) *Durrieu and Nelson* (2013). The circumstances to which the world has come together to fight an intergalactic war imply that the timeline on which resources can be utilized is at the forefront of our concern. Through our studies, we have determined that the fastest and most reasonable solution would be the harvesting of a near-earth asteroid that harbors a sufficient composition of metals for the galactic war. In short, the logistics of such an undertaking would require the immediate establishment of a Moon base as the estimated cost to reach the lunar orbit from Earth is at $30 000 USD per kilogram *Roberts and Kaplan* (2020). At this stage, a probe would be sent out to the asteroid to stabilize and redirect its orbital mechanics with vaporizing lasers such that it enters the Earth-Moon system within a year, and the mining process can begin.

5.1.4 The Construction

We have discussed the various elements in play with developing a predictive model for humanity's most effective military production in an intergalactic war. At this point in our project, we have implicated the following contributions to a nations armature rates:

The Mutual Fear Model

$$\begin{aligned} \frac{dx}{dt} &= ay \\ \frac{dy}{dt} &= bx \end{aligned} \tag{5.1}$$

where: a and b are simply constants that factor into the yearly rates $x(t)$ and $y(t)$ for two nations in an arms race.

Excessive Expenditure

Which altered the above Mutual Fear Model to present a realistic economical ceiling for the expenditure rates, expressed as:

$$\begin{aligned} \frac{dx}{dt} &= ay - mx \\ \frac{dy}{dt} &= bx - ny \end{aligned} \tag{5.2}$$

where: m and n are factors that result from a nations economy due to the uptick or reduction in military expenditure, these are subtracted from the overall armature rate implicating that a positive set of factors means the military expenditure is hindering the economy whereas negative parameters m and n mean the military expenditure has room to grow.

Sentimental Prospects

We incurred that nations have an evolving set of beliefs and that the perceptions on the state of the arms race would either accelerate military expenditure - when they have a mutual grievance, or subside military expenditure - when their exists a feeling of good will. This was the famous Richardson Model:

$$\begin{aligned}\frac{dx}{dt} &= ay - mx + r \\ \frac{dy}{dt} &= bx - ny + s\end{aligned} \tag{5.3}$$

where: r and s are contributing factors that directly increase or decrease expenditure rates when a nation is inclined to express their prejudice or represent conformity.

Surveillance

We then presented the conjecture on any two nations in an arms race finding it difficult to sit idly being supposedly oblivious or indifferent to what the other nation is boasting. We utilized a logistic version of the above Richardson Model such that a more realistic estimation on a nations maximum expenditure could be made. This maximum expenditure was then increased by a surveillance expenditure, in essence implicating a nations persistence to spy on the other nations arsenal by increasing the resultant ceiling on their own arsenal. The resultant model:

$$\begin{aligned}\frac{dx}{dt} &= a(1 - \frac{x}{A_p})y - mx + r \\ \frac{dy}{dt} &= b(1 - \frac{y}{A_g})x - ny + s\end{aligned} \tag{5.4}$$

where: $A_p = K_p + v_p$ represents the the Purple nation's expenditure ceiling as the addition on their maximum expenditure on armature (K_p) and their expenditure on surveillance (v_p). Likewise for Green, $A_g = K_g + v_g$.

Three Nation Model

Our final ingredient to the contributions on a nations armature rates, whilst in an arms race, was the prospective of a three nation arms race. This was represented as:

$$\begin{aligned}\frac{dx}{dt} &= m_x(\bar{x} - x) + r_{xy}y + r_{xz}z \\ \frac{dy}{dt} &= m_y(\bar{y} - y) + r_{yx}x + r_{yz}z \\ \frac{dz}{dt} &= m_z(\bar{z} - z) + r_{zy}y + r_{xz}x\end{aligned} \tag{5.5}$$

where: x, y and z represent the armature rates of each nation above their baseline rates (absolved of any external influence) as $\bar{x}, \bar{y}$ and $\bar{z}$. Further, any discrepancy from the usual armature rates for a nation is adjusted to the parameters m_x, m_y and m_z which would be synonymous to the parameters a and b from the two nation models above. Finally, there are the r_{ij} (for $i, j \;\varepsilon\; x, y, z$) parameters which define the external influences that affect a nations armature.

We will utilize the surveillance discussion above and the logistic variation to re-express the above three nation model as:

$$\begin{aligned} \frac{dx}{dt} &= m_x(\bar{x} - x) + (r_{xy}y + r_{xz}z)(1 - \frac{x}{A_x}) \\ \frac{dy}{dt} &= m_y(\bar{y} - y) + (r_{yx}x + r_{yz}z)(1 - \frac{y}{A_y}) \\ \frac{dz}{dt} &= m_z(\bar{z} - z) + (r_{zy}y + r_{xz}x)(1 - \frac{z}{A_z}) \end{aligned} \tag{5.6}$$

where: $A_x = K_x + v_x$, $A_y = K_y + v_y$ and $A_z = K_z + v_z$ all with synonymous definitions as before.

Parameterization

We now have a model that presents us with an opportunity to begin redevelopment to represent the most efficient global market (**5.1**). For simplicity we will present this discussion for nation x, and the remaining nations will simply follow:

- The armature rate (x) of any nation will be above their baseline rate ($\bar{x}$), implying $(\bar{x} - x) < 0$, however the overall term, $m_x(\bar{x} - x) > 0$, so $m_x < 0$.

- Due to the collaborative nature of all nations, their will be no need for surveillance, so the ceiling on a given nations inherent military expenditure will be elevated. This means $A_z \to K_z$.

- The collective fear of alien weaponry means an extraordinary resentment towards the galactic species (G_x), and any deterrence to a continued growth will be a result of limited resources (R_x). This can be expressed as an integrated competition term $(G_x x - R_x xyz)$ in place of the international influences term $(r_{xy}y + r_{xz}z)$.

- The only reduction on a nations armature rate, the resource limitation $R_x xyz$, must reach a stability that minimizes its impact, $\lim_{t\to 0} R_x xyz \to 0$ (**5.2**).

- The self-replicating process of delivering materials to Earth through 3D printing as more material is made available, allows us to substitute the limited resources term $R_x xyz$ with a time-dependent exponential decay $xyze^{-R_x t}$ (**5.3**). Implying the limitations of available resources is exponentially decreasing, or the availability of resources is exponentially increasing.

The Combined Model

Utilizing the above parameterization on the most recent model above, we can finally present our model:

$$\begin{aligned}\frac{dx}{dt} &= m_x(\bar{x}-x)+(G_x x - xyze^{-R_x t})(1-\frac{x}{K_x})\\ \frac{dy}{dt} &= m_y(\bar{y}-y)+(G_y y - xyze^{-R_y t})(1-\frac{y}{K_y})\\ \frac{dz}{dt} &= m_z(\bar{z}-z)+(G_z z - xyze^{-R_z t})(1-\frac{z}{K_z})\end{aligned} \tag{5.7}$$

where: $m_i < 0$ and $G_i > 0$ for $i = x, y, z$.

5.2 Solution

The set of equations as described above in (7) are exceedingly complex and with the motivation to realize the predictive power of our model, we will embrace a series of plausible simplifications that are detailed further in (**6.4.2**).

In short, after the substitution of (+1) for the Gs and (-1) for the ms, we are able to utilize the nature of negative exponentials to associate a value that is within a quarter of the resource usage each of these nations endure past the first year of production. Given the timeframe on armature production with limitless resources is the year it takes to begin the mining process, we reduce the exponential terms to a constant that resembles the consistent resource depletion after the first year of production. Further, the cross-proportionality of competition is normalized to the same set of parameters, as the abundance of resources would resemble an absence of competition that sets any two nations apart.

5.2.1 The (Simplified) Combined Model

We have now reduced our model to its following simplified form:

$$\begin{aligned}\frac{dx}{dt} &= (a)x^2+(bk_x)x-(k_x\bar{x})\\ \frac{dy}{dt} &= (a)y^2+(bk_y)x-(k_y\bar{y})\\ \frac{dz}{dt} &= (a)z^2+(bk_z)x-(k_z\bar{z})\end{aligned} \tag{5.8}$$

where:

$$a = (yze^{-R_x t}-1) = (xze^{-R_y t}-1) = (xye^{-R_z t}-1),$$
$$b = (1-yze^{-R_x t}) = (1-xze^{-R_y t}) = (1-xye^{-R_z t}),$$
$$m_{x,y,z} = m = -1, \text{ and } G_{x,y,z} = G = +1.$$

It is simply the case that a must be a positive parameter and b must be a negative parameter as a result of the logistic economical factors we prescribed in (**5.1**).

We set $a = 2$, $b = -2$ and assume the baseline rates for all nations is one. Given the suggested process by which limitless resources are made available shortly after the first year of this global initiative, we are justified in assuming the armature rates will have doubled for each nation at the conclusion of the second year. In other words, we have the following set of initial conditions: $x(2) = y(2) = z(2) = 2$. We can now solve the above model for its general solutions, representative of each nations individual armature rates over time.

$$\begin{aligned} x(t) &= \frac{1}{2}(\sqrt{k_x+2}\sqrt{k_x}\,tanh[-\sqrt{k_x}\sqrt{k_x+2} - tanh^{-1}(\frac{k_x-4}{\sqrt{k_x}\sqrt{k_x+2}})] + k_x) \\ y(t) &= \frac{1}{2}(\sqrt{k_y+2}\sqrt{k_y}\,tanh[-\sqrt{k_y}\sqrt{k_y+2} - tanh^{-1}(\frac{k_y-4}{\sqrt{k_y}\sqrt{k_y+2}})] + k_y) \\ z(t) &= \frac{1}{2}(\sqrt{k_z+2}\sqrt{k_z}\,tanh[-\sqrt{k_z}\sqrt{k_z+2} - tanh^{-1}(\frac{k_z-4}{\sqrt{k_z}\sqrt{k_z+2}})] + k_z) \end{aligned} \tag{5.9}$$

The above general solutions to (8) were solved using Wolfram Mathematica.

An arbitrary set of parameters can now be assigned for the maximal economic ceilings for each nation and the predictive nature of our model can be visualized.

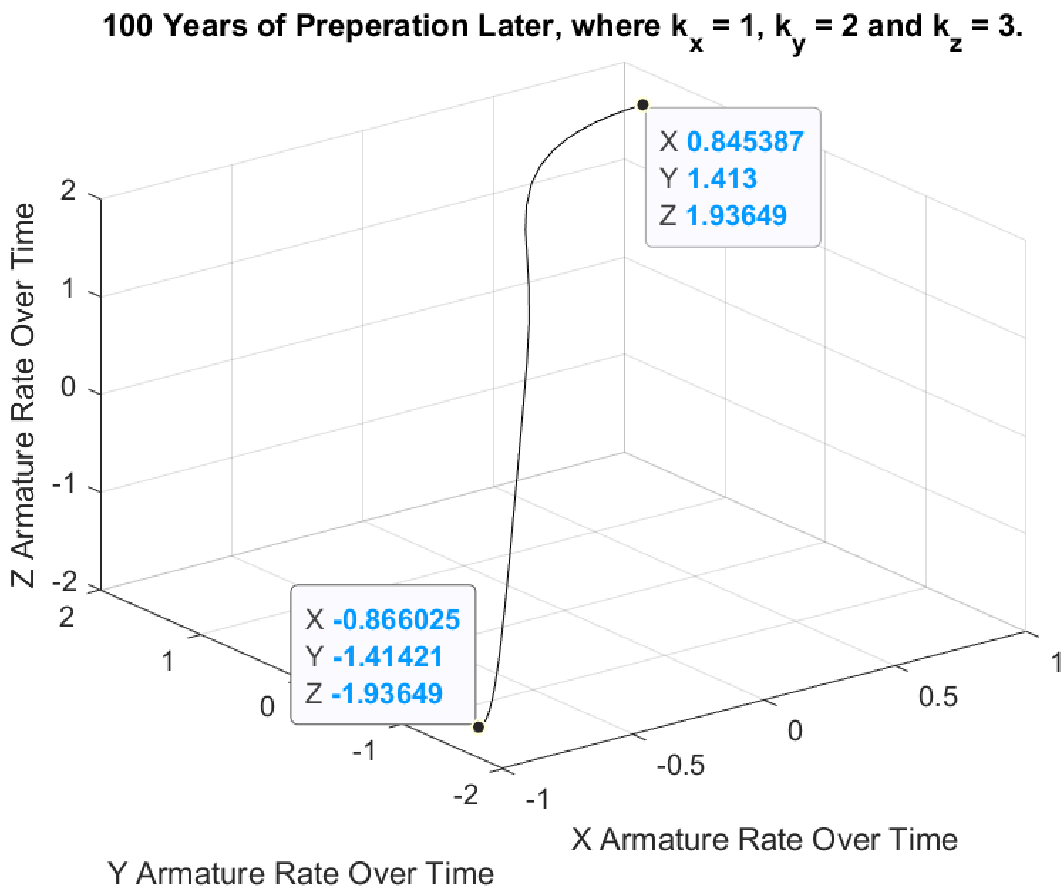


*Figure 5.1: The supplementary code for the above figure can be found in (**6.4.2**).*

5.3 Interpretation

The immediate observation from the visualization of how armature rates grow over time is the negative nature of the curve for various periods. Our intuition would have us believe that given the objective of producing the most extensive arsenal, each nation's armature rates would grow indefinitely. Yet, Figure 1 shows a fluctuation between increasing and decreasing rates. This is best understood from the perspective of the global market that was built on how much more a nation can spend than another. These nations are coming right out of a heated race that artificially elevated their expenditure rates due to competition. When that competition is removed, both in terms of resources and arsenal size, a nation will seek to establish an armature rate that is elevated above their baseline rate and sustainable for the long run, as we see in our 100-year simulation. Further, a nation is not inclined to sacrifice its individual economies to pursue any advancements ineffective production or better weaponry. The collaborative nature of the global economy, and our understandings from ($\mathbf{5.1}-\mathbf{5.2}$), imply that as more time is given for research and development on military weapons, the inevitable outcome on the necessary expenditure to achieve the same levels of production as before, will be reduced given a natural tendency towards efficiency.

These are fascinating developments from our all but limited collection of impactful components, which are relatively rudimentary in nature - aspects that do not have a great deal of judgment. We will address this aspect of game theory which has been overlooked throughout in a moment. Still, before that, we look to address the concerns of the circumstances we have endeavored to create this model - the galactic war. In short, a galactic species with the capability of going to war with another species in a different galaxy is referred to as a *Type Three Civilization* on the Kardashev scale *Kardashev* (1964). This implies that this species can control energy at the scale of its host galaxy. The human race, in this era, is on a transition from controlling the energy available on its host planet (*Type One*) to the energy at the scale of its planetary system (*Type Two*). In other words, due to the simple fact of the energy that is attainable for each civilization, the galactic species will have no problem doing what they want when it comes to the human race. The study of what the human race is capable of on a destructive level is a valuable measure of the human race's potential within a given era that is otherwise being utilized to compete with one another. This project offers a valuable opportunity to reflect on the human race and what truly matters to those fortunate enough to be the decision-makers; the further discussion will venture outside the scope of this report. So the reader is left to their judgments.

5.4 Critique

The inevitably unpredictable nature of one's judgment brings complexity to the studies of strategic interactions amongst rational decision-makers or game theory. The most critical of the criticisms that can be made for the development of our model is the lack of any capacity that the nations have to form their own opinions or evaluations. We assumed the reactive nature in the fear-mongering models was to respond with more money, with no further conjecture on where or how that money is best spent. In warfare, the positioning of military troops, the establishment of remote bases of operation, and the advancements in long-range weaponry and reconnaissance represent only a handful of aspects that directly affect how a nation perceives its state in the power struggle. At the national level, the decision-makers have their country's interests at the forefront of their concerns; one of these interests is the quality of life of its people. War is a taxing burden on society; it impacts the decisions people make on how they choose to invest their time and efforts daily. If the latest speculations on what the opposition has developed uncover a capability for nuclear destruction, the fear instilled in society will drive it to invest in infrastructures such as nuclear bomb shelters. This directly impacts the nations' economy as the global market will shift to meet the demand for the resources needed to facilitate societal directions. These are observations that arise at the foundations of this model. The absence of any implementations that address these concerns is only multiplied and further ingrained as we develop models under the same set of assumptions. In our model's final state, we have the psychological factors that are overlooked but logistical ones that arise from the leaps made to accomplish a predictive model. One such aspect breaks down the line of reasoning that was expanded on in the preceding section, how effective are the pre-existing military arsenals in space? If the answer is anything short of "just as effective", each nation's armature rates will stagnate for some time as the collective effort will look to develop weaponry that can be used in space. Further, the weaponry would be at a stage of effectiveness that would be synonymous with the most primitive weaponry through human existence (as a result of the natures of space). This transition to more effective and reasonable weaponry will drastically impact the rates of expenditure nations will undertake on their military sectors. The collaborative nature could perhaps conclude that one nation focuses on establishing effective means of transport for soldiers in space, for example, while the others look to develop the weapons these soldiers can use. Such a shift in market share for aerospace research and development for one nation, which was previously a part of equilibrium between all nations in the market of military research and development, results in drastic changes to how the new equilibrium's for effective allocations of resources (**5**.**1**) is re-established. Such a phenomenon is so far removed from the state of our current model that the improvements to be made in this model begin from the very start.

We have learned from this experience in the creation and alterations of models that the predicament in trying to model large-scale behaviors and the exhaustive accommodation of all plausible factors in play. At the scale of an arms race, there are too many factors that are inexplicably hard to parameterize. Any motivation for large-scale models should be delegated into smaller relationships that comprise the larger system. In conclusion, this was an invaluable experience in attempting to describe a complex system using mathematical concepts and language, we hope our discussions have enlightened the reader on the process of developing more accurate and elegant mathematical models.

Chapter 6

Appendix

6.1 Mutual Fear Model

6.1.1 Geogebra

Phase spaces are used to analyze autonomous differential equations. The two dimensional case is specially relevant, because it is simple enough to give us lots of information just by plotting it.

6.1.2 MATLAB Code

```
syms x(t) y(t)
a = 1;
b = 1;
ode1 = diff(x) == a*y;
ode2 = diff(y) == b*x;
odes = [ode1; ode2]
S = dsolve(odes)
xSol(t) = S.x
ySol(t) = S.y
cond1 = x(0) == 0;
cond2 = y(0) == 0;
conds = [cond1; cond2];
[xSol(t), ySol(t)] = dsolve(odes,conds)
fplot(xSol)
hold on
fplot(ySol)
grid on
legend('xSol','ySol','Location','best')
```

6.2 Richardson Model

6.2.1 MATLAB Code - ODE System Solver

```
% 2 System ode Solver and Plotter
syms x(t) y(t)
a = 1;
b = 1;
m = 0.5;
n = 0.5;
r = 1;
s = 1;
ode1 = diff(x) == a*y - m*x + r;
ode2 = diff(y) == b*x - n*y + s;
odes = [ode1; ode2]
S = dsolve(odes)
xSol(t) = S.x
ySol(t) = S.y
cond1 = x(0) == 0;
cond2 = y(0) == 1;
conds = [cond1; cond2];
[xSol(t), ySol(t)] = dsolve(odes,conds)
fplot(xSol)
hold on
fplot(ySol)
grid on
legend('xSol','ySol','Location','best')
```

6.3 Three Nation Model

6.3.1 MATLAB Code - Phase Potrait

```
% Case 1 : m > r > 0
% Case 2 : r > m > 0
% Case 3 : |r_{ij}|> m > 0
% Now we consider r_{zx}, r_{zy} < 0 and the others to be
    positive
% Case 5 :|r_{ij}|> m
% Now we consider r_{yx}, r_{xy} < 0 and the others to be
    positive
% Case 5 is considered here as an example
m = 0.05;
r1 = -0.1; r2 = 0.1; r3 = -0.1; r4 = 0.1; r5 = 0.1; r6 =
   0.1;
m1 = 0.05;
```

```
m2 = 0.05;
m3 = 0.05;
dxdt = @(x,y,z) m.*(m1 - x) + r1.*y + r2.*z;
dydt = @(x,y,z) m.*(m2 - y) + r3.*x + r4.*z;
dzdt = @(x,y,z) m.*(m3 - z) + r5.*x + r6.*y;
x = -10:2:10;
y = -10:2:10;
z = -10:2:10;
[X,Y,Z] = meshgrid(x,y,z);
dX = dxdt(X,Y,Z);
dY = dydt(Z,Y,Z);
dZ = dzdt(X,Y,Z);
quiver3(X,Y,Z,dX,dY,dZ);
axis tight
```

6.3.2 MATLAB Code - ODE System Solver

```
% 3 System ode Solver and Plotter
syms x(t) y(t) z(t)
ode1 = diff(x) == m.*(m1 - x) + r1.*y + r2.*z;
ode2 = diff(y) == m.*(m2 - y) + r3.*x + r4.*z;
ode3 = diff(z) == m.*(m3 - z) + r5.*x + r6.*y;
odes = [ode1; ode2; ode3]
S = dsolve(odes)
xSol(t) = S.x
ySol(t) = S.y
zSol(t) = S.z
cond1 = x(0) == 0;
cond2 = y(0) == 1;
cond3 = z(0) == 2;
conds = [cond1; cond2; cond3];
[xSol(t), ySol(t), zSol(t)] = dsolve(odes,conds)
fplot(xSol)
hold on
fplot(ySol)
hold on
fplot(zSol)
grid on
legend('xSol','ySol','zSol','Location','best')
```

6.3.3 Eigenvalues

```
% Eigen values
J = [-m r1 r2; r3 -m r4; r5 r6 -m]
D = eig(J)
```

$$
\begin{aligned}
&\lambda_1 = \\
&\frac{1}{3\sqrt[3]{2}}((-2 m1^{3} + 3 m1^{2} m2 + 3 m1^{2} m3 + \sqrt{}((-2 m1^{3} + 3 m1^{2} m2 + 3 m1^{2} m3 + \\
&\qquad 3 m1 m2^{2} - 12 m1 m2 m3 + 3 m1 m3^{2} - 9 m1 r1 r3 - \\
&\qquad 9 m1 r2 r5 + 18 m1 r4 r6 - 2 m2^{3} + 3 m2^{2} m3 + \\
&\qquad 3 m2 m3^{2} - 9 m2 r1 r3 + 18 m2 r2 r5 - \\
&\qquad 9 m2 r4 r6 - 2 m3^{3} + 18 m3 r1 r3 - 9 m3 r2 r5 - \\
&\qquad 9 m3 r4 r6 + 27 r1 r4 r5 + 27 r2 r3 r6)^{2} + \\
&\quad 4 (3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - \\
&\qquad (m1 + m2 + m3)^{2})^{3}) + \\
&\quad 3 m1 m2^{2} - 12 m1 m2 m3 + 3 m1 m3^{2} - 9 m1 r1 r3 - \\
&\quad 9 m1 r2 r5 + \\
&\quad 18 m1 r4 r6 - \\
&\quad 2 m2^{3} + \\
&\quad 3 m2^{2} m3 + \\
&\quad 3 m2 m3^{2} - \\
&\quad 9 m2 r1 r3 + \\
&\quad 18 m2 r2 r5 - \\
&\quad 9 m2 r4 r6 - \\
&\quad 2 m3^{3} + 18 m3 r1 r3 - \\
&\quad 9 m3 r2 r5 - 9 m3 r4 r6 + \\
&\quad 27 r1 r4 r5 + \\
&\quad 27 r2 r3 r6) \wedge (1/3)) - \\
&(\sqrt[3]{2} (3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - (m1 + m2 + m3)^{2})) /
\end{aligned}
$$

$$
\begin{aligned}
&(\sqrt[3]{2} (3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - (m1 + m2 + m3)^{2})) / \\
&(3 \\
&\quad (-2 m1^{3} + 3 m1^{2} m2 + 3 m1^{2} m3 + \\
&\qquad \sqrt{}((-2 m1^{3} + 3 m1^{2} m2 + 3 m1^{2} m3 + 3 m1 m2^{2} - 12 m1 m2 m3 + \\
&\qquad\quad 3 m1 m3^{2} - 9 m1 r1 r3 - 9 m1 r2 r5 + 18 m1 r4 r6 - 2 \\
&\qquad\quad m2^{3} + 3 m2^{2} m3 + 3 m2 m3^{2} - 9 m2 r1 r3 + 18 m2 \\
&\qquad\quad r2 r5 - 9 m2 r4 r6 - 2 m3^{3} + 18 m3 r1 r3 - 9 m3 \\
&\qquad\quad r2 r5 - 9 m3 r4 r6 + 27 r1 r4 r5 + 27 r2 r3 r6)^{2} + \\
&\qquad 4 (3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - \\
&\qquad\quad (m1 + m2 + m3)^{2})^{3}) + 3 m1 m2^{2} - \\
&\qquad 12 m1 m2 m3 + 3 m1 m3^{2} - 9 m1 r1 r3 - 9 m1 r2 r5 + \\
&\qquad 18 m1 r4 r6 - 2 m2^{3} + 3 m2^{2} m3 + 3 m2 m3^{2} - 9 m2 r1 r3 + \\
&\qquad 18 m2 r2 r5 - 9 m2 r4 r6 - 2 m3^{3} + 18 m3 r1 r3 - \\
&\qquad 9 m3 r2 r5 - 9 m3 r4 r6 + 27 r1 r4 r5 + 27 r2 r3 r6) \wedge \\
&\quad (1/3)) + \frac{1}{3} (-m1 - m2 - m3)
\end{aligned}
$$

Figure 6.1: Eigenvalue 1

$$
\begin{aligned}
&\lambda_2 = \\
&-\frac{1}{6\sqrt[3]{2}}(1 - i\sqrt{3})(-2 m1^{3} + 3 m1^{2} m2 + 3 m1^{2} m3 + \sqrt{}((-2 m1^{3} + 3 m1^{2} m2 + \\
&\qquad 3 m1^{2} m3 + 3 m1 m2^{2} - 12 m1 m2 m3 + 3 m1 m3^{2} - \\
&\qquad 9 m1 r1 r3 - 9 m1 r2 r5 + 18 m1 r4 r6 - 2 m2^{3} + \\
&\qquad 3 m2^{2} m3 + 3 m2 m3^{2} - 9 m2 r1 r3 + 18 m2 r2 r5 - \\
&\qquad 9 m2 r4 r6 - 2 m3^{3} + 18 m3 r1 r3 - 9 m3 r2 r5 - \\
&\qquad 9 m3 r4 r6 + 27 r1 r4 r5 + 27 r2 r3 r6)^{2} + \\
&\quad 4 (3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - \\
&\qquad (m1 + m2 + m3)^{2})^{3}) + \\
&\quad 3 m1 m2^{2} - 12 m1 m2 m3 + 3 m1 m3^{2} - 9 m1 r1 r3 - \\
&\quad 9 m1 r2 r5 + \\
&\quad 18 m1 r4 r6 - \\
&\quad 2 m2^{3} + \\
&\quad 3 m2^{2} m3 + \\
&\quad 3 m2 m3^{2} - \\
&\quad 9 m2 r1 r3 + \\
&\quad 18 m2 r2 r5 - 9 m2 r4 r6 - \\
&\quad 2 m3^{3} + 18 m3 r1 r3 - \\
&\quad 9 m3 r2 r5 - 9 m3 r4 r6 + \\
&\quad 27 r1 r4 r5 + 27 r2 r3 r6) \wedge (1/3) + \\
&((1 + i\sqrt{3})(3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - \\
&\quad (m1 + m2 + m3)^{2})) /
\end{aligned}
$$

$$
\begin{aligned}
&\quad 3 m1 m2^{2} - 12 m1 m2 m3 + 3 m1 m3^{2} - 9 m1 r1 r3 - \\
&\quad 9 m1 r2 r5 + \\
&\quad 18 m1 r4 r6 - \\
&\quad 2 m2^{3} + \\
&\quad 3 m2^{2} m3 + \\
&\quad 3 m2 m3^{2} - \\
&\quad 9 m2 r1 r3 + \\
&\quad 18 m2 r2 r5 - 9 m2 r4 r6 - \\
&\quad 2 m3^{3} + 18 m3 r1 r3 - \\
&\quad 9 m3 r2 r5 - 9 m3 r4 r6 + \\
&\quad 27 r1 r4 r5 + 27 r2 r3 r6) \wedge (1/3) + \\
&((1 - i\sqrt{3})(3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - \\
&\quad (m1 + m2 + m3)^{2})) / \\
&(3 \times 2^{2/3} (-2 m1^{3} + 3 m1^{2} m2 + 3 m1^{2} m3 + \\
&\qquad \sqrt{}((-2 m1^{3} + 3 m1^{2} m2 + 3 m1^{2} m3 + 3 m1 m2^{2} - 12 m1 m2 m3 + \\
&\qquad\quad 3 m1 m3^{2} - 9 m1 r1 r3 - 9 m1 r2 r5 + 18 m1 r4 r6 - 2 \\
&\qquad\quad m2^{3} + 3 m2^{2} m3 + 3 m2 m3^{2} - 9 m2 r1 r3 + 18 m2 \\
&\qquad\quad r2 r5 - 9 m2 r4 r6 - 2 m3^{3} + 18 m3 r1 r3 - 9 m3 \\
&\qquad\quad r2 r5 - 9 m3 r4 r6 + 27 r1 r4 r5 + 27 r2 r3 r6)^{2} + \\
&\qquad 4 (3 (m1 m2 + m1 m3 + m2 m3 - r1 r3 - r2 r5 - r4 r6) - \\
&\qquad\quad (m1 + m2 + m3)^{2})^{3}) + 3 m1 m2^{2} - \\
&\qquad 12 m1 m2 m3 + 3 m1 m3^{2} - 9 m1 r1 r3 - 9 m1 r2 r5 + \\
&\qquad 18 m1 r4 r6 - 2 m2^{3} + 3 m2^{2} m3 + 3 m2 m3^{2} - \\
&\qquad 9 m2 r1 r3 + 18 m2 r2 r5 - 9 m2 r4 r6 - \\
&\qquad 2 m3^{3} + 18 m3 r1 r3 - 9 m3 r2 r5 - \\
&\qquad 9 m3 r4 r6 + 27 r1 r4 r5 + 27 r2 r3 r6) \wedge \\
&\quad (1/3)) + \frac{1}{3} (-m1 - m2 - m3)
\end{aligned}
$$

Figure 6.2: Eigenvalues 2 and 3

6.3.4 Wolfram Alpha - Eigenvalues

The figures above are the eigenvalues for the 3 nation model. These are the eigenvalues when $m1 = m_x$, $m2 = m_y$, $m3 = m_z$ and $r1 = r_{xy}$, $r2 = r_{xz}$, $r3 = r_{yx}$, $r4 = r_{yz}$, $r5 = r_{zx}$, $r6 = r_{zy}$. These eigenvalues are found using Wolfram alpha.

6.4 Combined Model

6.4.1 The simplification of The Combined Model

We begin with **The Combined Model**:

$$\begin{aligned}\frac{dx}{dt} &= m_x(\bar{x}-x)+(G_x x - xyze^{-R_x t})(1-\frac{x}{K_x})\\ \frac{dy}{dt} &= m_y(\bar{y}-y)+(G_y y - xyze^{-R_y t})(1-\frac{y}{K_y})\\ \frac{dz}{dt} &= m_z(\bar{z}-z)+(G_z z - xyze^{-R_z t})(1-\frac{z}{K_z})\end{aligned} \tag{6.1}$$

and let $m_i = m = -1$ and $G_i = G = 1$,

$$\begin{aligned}\frac{dx}{dt} &= (x-\bar{x})+x(1-yze^{-R_x t})(1-\frac{x}{K_x})\\ \frac{dy}{dt} &= (y-\bar{y})+y(1-xze^{-R_y t})(1-\frac{y}{K_y})\\ \frac{dz}{dt} &= (z-\bar{z})+z(1-xye^{-R_z t})(1-\frac{z}{K_z}).\end{aligned} \tag{6.2}$$

Expanding the above and rearranging in leading powers of x, y and z, we get

$$\begin{aligned}\frac{dx}{dt} &= \frac{x^2}{K_x}(yze^{-R_x t}-1)+x(1-yze^{-R_x t})-\bar{x}\\ \frac{dy}{dt} &= \frac{y^2}{K_y}(xze^{-R_y t}-1)+y(1-xze^{-R_y t})-\bar{y}\\ \frac{dz}{dt} &= \frac{z^2}{K_z}(xye^{-R_z t}-1)+z(1-xye^{-R_z t})-\bar{z}\end{aligned} \tag{6.3}$$

which is the expanded form of

$$\begin{aligned}\frac{dx}{dt} &= (a)x^2+(bk_x)x-(k_x\bar{x})\\ \frac{dy}{dt} &= (a)y^2+(bk_y)x-(k_y\bar{y})\\ \frac{dz}{dt} &= (a)z^2+(bk_z)x-(k_z\bar{z})\end{aligned} \tag{6.4}$$

where:

$$\begin{gathered} a = (yze^{-R_x t}-1) = (xze^{-R_y t}-1) = (xye^{-R_z t}-1),\\ b = (1-yze^{-R_x t}) = (1-xze^{-R_y t}) = (1-xye^{-R_z t}),\\ m_{x,y,z} = m = -1, \text{ and } G_{x,y,z} = G = +1.\end{gathered}$$

6.4.2 MATLAB code for Figure 1

```
clc;
clear;

t = 1:(1/12):100;

[k1, k2, k3] = deal(1, 2, 3);

xt1 = 0.5*(sqrt(k1+2)*sqrt(k1)*tanh((-sqrt(k1)*sqrt(k1+2)*t)+(2*sqrt(k1)*sqrt(k1+2))-(tanh((k1-4)/(sqrt(k1)*sqrt(k1+2))))^-1) + k1));
yt1 = 0.5*(sqrt(k2+2)*sqrt(k2)*tanh((-sqrt(k2)*sqrt(k2+2)*t)+(2*sqrt(k2)*sqrt(k2+2))-(tanh((k2-4)/(sqrt(k2)*sqrt(k2+2))))^-1) + k2));
zt1 = 0.5*(sqrt(k3+2)*sqrt(k3)*tanh((-sqrt(k3)*sqrt(k3+2)*t)+(2*sqrt(k3)*sqrt(k3+2))-(tanh((k3-4)/(sqrt(k3)*sqrt(k3+2))))^-1) + k3));

plot3(xt1,yt1,zt1, color = "black")
title("100 Years of Preperation Later, where k_x = 1, k_y = 2 and k_z = 3.")
xlabel("X Armature Rate Over Time")
ylabel("Y Armature Rate Over Time")
zlabel("Z Armature Rate Over Time")
grid on
```

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