Chapter 7

Zeta(2k)

How do we dervive the $\zeta(2k)$ where $k \in \mathbb{N}$ using fourier analysis?

Introduction

Recap: We start with some periodic function.

Periodic :$f(x+p) = f(x)\forall x$ where p is fixed called a period of f(x). A continous periodic function can be expressed as an infinite series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{2\pi nx}{p}) + b_n sin(\frac{2\pi nx}{p})$$

where

$$a_n = \frac{2}{p}\int_I f(x)cos(\frac{2\pi nx}{p})dx(n \geq 0)$$

and

$$a_n = \frac{2}{p}\int_I f(x)sin(\frac{2\pi nx}{p})dx(n \geq 0)$$

where I is any interval of length p.

Bernoulli polynomials

Usual definition: Lets look at the case of the usual way how the bernoulli's polynomail is defined

$$\frac{(te^{xt})}{(e^t - 1)} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

Question : We might wonder why is this a good description of the bernoulis number.

Another ideas is to work towards another description of the $B_n(x)$

- Let

$$p_n(x) = \frac{1}{n!}B_n(x)$$

then

$$\frac{(te^{xt})}{(e^t - 1)} = \sum_{n=0}^{\infty} p_n(x)t^n$$

Proposition 1:

- $p_o(x) = 1$
- $p_{n+1}(x)' = p_n(x))\forall n \geq 0$
- $p_n(0) = p_n(1)\forall n \geq 2$

Proof:

- 1) This is clear from the defining relation
- 2) We differentiate the defining relation with respect to x:

$$\frac{(t^2e^{xt})}{(e^t - 1)} = \sum_{n=0}^{\infty} p_n'(x)t^n$$

$$= \sum_{n=1}^{\infty} p_n'(x)t^n$$

because $p'(x) = 0$. Dividing both sides by t and using the defining relation again, we obtain

$$\sum_{n=0}^{\infty} p_n(x)t^n = \sum_{n=1}^{\infty} p_n'(x)t^{n-1}$$

$$= \sum_{n=0}^{\infty} p_{n+1}'(x)t^n$$

so $p_{n+1}(x)' = p_n(x))\forall n \geq 0$

- 3)

$$\sum_{n=0}^{\infty} p_n(1)t^n = \frac{(te^t)}{(e^t-1)}$$
$$= \frac{(te^t-1)+t}{(e^t-1)}$$
$$= t + \frac{t}{(e^t-1)}$$
$$= t + \sum_{n=0}^{\infty} p_n(0)t^n$$

  But we can do better that is to change it i.e:

- $p_o(x) = 1$
- $p_{n+1}(x)' = p_n(x))\forall n \geq 0$
- $\int_0^1 p_n(x)dx = 0 \forall n \geq 1$

  Note also that it is obvious that properties (1)', (2)', and (3)' define a family of polynomials $p_n(x)_{n\geq 0}$. One could therefore use these properties to define the $p_n(x)$.

Proposition 2:

That is $\forall n \geq 0$

$$p_n(1-x) = (-1)^n(p_n(x))$$

Proof

$$\sum_{n=0}^{\infty} p_n(1-x)t^n = \frac{(te^{(1-x)t})}{(e^t-1)}$$
$$= \frac{(te^t e^{-xt})}{(e^t-1)}$$
$$= \frac{(-ue^{-u}e^{ux})}{(e^{-u}-1)}$$

where u = -t

$$= \frac{(ue^{ux})}{(e^u-1)}$$

$$= \sum_{n=0}^{\infty} p_n(x) u^n$$

$$= \sum_{n=0}^{\infty} (-1)^n p_n(x) t^n$$

Reminder : $B_n(x) = n! p_n(x)$ scaled to be monic

Let $f_n : \mathbb{R} \to \mathbb{R}$ be the periodic function of period 1 agreeing with the polynomial $p_n(x)$ on [0, 1). Note that, if $n \geq 2$, then

$$p_n(1) - p_n(0) = \int_0^1 p_{n-1}(x)dx = 0$$

so f_n is continuous on $\mathbb{R}$.

We will compute the fourier series of $p_n(x)$ where n is even .More precisely we extend $p_n(x)$ to a periodic function f_n such that

$$f_n(x) = p_n(x)$$

if $0 \leq x < 1$,

$$f_n(x+1) = f_n(x) \forall x \in R$$

if

$$n \neq 1, f_n(0) = p_n(0) = p_n(1) = f_n(1)$$

therefore f_n is continuous.

Lemma 3

If $n \geq 0$ is even, then f_n is an even function.

Proof :

Let $\{a\} = a - \lfloor a \rfloor$ denote the fractional part of a real number a, and note that $\{-a\} = 1$ - $\{a\}$ if $a \in \mathbb{R} - \mathbb{Z}$. Then if $x \in \mathbb{R} - \mathbb{Z}$,

$$f_n(-x) = p_n(\{-x\})$$

$$= p_n(\{1 - x\})$$

$$= p_n(\{x\})$$

by Proposition 2

$$= f_n(x)$$

Fourier Series

Let the fourier series of f_n be

$$\frac{a_{n,0}}{2} + \sum_{m=1}^{\infty}(a_{n,m}cos(2\pi mx) + b_{n,m}sin(2\pi mx))$$

Lets calculate some terms

$$a_{n,0} = 2\int_0^1 f_n(x)dx = 2\int_0^1 p_n(x)dx = 0\forall n \geq 1$$

Assume now that $n \geq 4$ is even . If $m \geq 1$

$$a_{n,m} = 2\int_0^1 f_m(x)cos(2\pi mx)dx =$$

$$2\int_0^1 p_n(x)cos(2\pi mx)dx =$$

$$\frac{2}{2\pi m}[p_n(x)sin(2\pi mx)]\Big|_0^1 - \frac{2}{2\pi m}\int_0^1 p_{n-1}(x)sin(2\pi mx)dx =$$

$$\frac{-2}{2\pi m} - [\frac{1}{2\pi m}[p_{n-1}(x)cox(2\pi mx)]\Big|_0^1 + \int_0^1 \frac{1}{2\pi m}p_{n-2}(x)cos(2\pi mx)dx =$$

$$2(\frac{-1}{2\pi m}^2)\int_0^1 p_{n-2}(x)cos(2\pi mx)dx =$$

$$2(\frac{1}{2\pi im}^2)\int_0^1 p_{n-2}(x)cos(2\pi mx)dx$$

Note that we introduce i here because it is algebraically expedient to do so, not because it is necessary.

By induction together with the fact that for all even $n \geq 2$

$$\int_0^1 p_n(x)cos(2\pi mx)dx$$

$$= \frac{1}{(2\pi im)^{n-2}}\left(\frac{1}{2\pi m})[p_2(x)sin(2\pi mx)]\Big|_0^1 - \frac{1}{2\pi m}\int_0^1 p_1(x)sin(2\pi mx)dx\right)$$

$$= \frac{1}{(2\pi im)^{n-2}}\left(\frac{-1}{2\pi m}\right)\left(\frac{-1}{2\pi m})[p_1(x)cos(2\pi mx)]\Big|_0^1 + \frac{1}{2\pi m}\int_0^1 cos(2\pi mx)dx\right)$$

$$= \frac{1}{(2\pi im)^{n-2}}\left(\frac{-1}{2\pi m}\right)\left(\frac{-1}{2\pi m} + 0\right)$$

$$= \frac{1}{(2\pi im)^n}$$

Thus

$$\int_0^1 p_2(x)cos(2\pi mx)dx = -(\frac{1}{2\pi im}^2)$$

$$a_{n,m} = \frac{-2}{(2\pi im)^n}$$

Now $a_{n,m} = 0$ when n is even and when $n \geq 4$ and $m \geq 1$, and the equality holds when n = 2 as well. The remaining case where n = 2 and m = 0 is immediate from the fact that $p_n(0) = p_n(1)$ when $n \geq 2$; specifically, $a_{n,0} = 0$. Hence, when $n \geq 2$ is even. By fourier convergence

$$f_n(x) = \sum_{m=1}^{\infty} a_{n,m}cos(2\pi mx) =$$

$$\sum_{m=1}^{\infty} \frac{-2}{(2\pi i)^n}\frac{cos(2\pi mx)}{m^n}$$

Hence now, as we have already remarked, f_n is continuous when $n \neq 1$, so we may evaluate the Fourier series at x = 0 to find that

$$f_n(0) = \sum_{m=1}^{\infty} \frac{-2}{(2\pi_i m)^n}$$

as $\cos(0) = 1$

$$\frac{-2}{(2\pi i)^n} \sum_{m=1}^{\infty} \frac{1}{m^n} = \frac{-2}{(2\pi i)^n}\zeta(n)$$

Rearranging gives us

$$\zeta(n) = \frac{-1}{2}(2\pi i)^n f_n(0) =$$

$$\frac{-1}{2}(2\pi i)^n p_n(0) =$$

$$\frac{-1}{2}(2\pi i)^{2k} \frac{B_{2k}}{2k!} =$$

$$\frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k!)}$$

where $k = \frac{n}{2}$.

Hence Done

QED