Solution space of a Homogeneous Linear Differential Equation

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A project submitted in fulfillment of the requirements for the course MATH 336(Honors Ordinary Differential Equations)

April 23, 2021

Acknowledgements

I thank God for blessing me to take this course as well as Professor Xinwei Yu for teaching this wonderful course and also thank my parents and friends for supporting me through this pandemic and helping me in every way possible

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Edmonton, Canada, 23/04/2021

Abstract

This project proves the following Theorem regarding the Solution Space of a Homogeneous Linear Differential Equation in the Complex Plane. All the required prerequisites/lemmas and additional Theorems are proved in order to prove the required theorem.

Theorem 1 *For any arbitrary open and connected region $R \subset \mathbb{C}$, The solution space of the homogeneous linear differential equation of order n*

$$y^{(n)}(z) + a_{n-1}(z)y^{(n-1)}(z) + \cdots + a_0(z)y(z) = 0$$

where every coefficient $a_j(z), j = 0, 1, 2, \ldots, n-1$, is continuous is n -dimensional ($\dim(V_R^n) = n$) if and only if every coefficient $a_j(z), j = 0, 1, 2, \ldots, n-1$ is analytic.

Contents

Chapter 1

Preliminaries

1.1 Analysis Background

As with polynomials, analytic functions can have repeated roots, and these are observed using derivatives. In the first section we will go ahead and prove few lemmas/theorem's and state definitions that can help us prove the required theorem. Unless stated all domains D are in $\mathbb{C}$. *Kuttler* (2020)

1.1.1 Formal Power Series

A (formal) power series centered at $x_0 \in \mathbb{C}$ is a sequence $a_n\,(n \in \mathbb{N}_0)$, written as $\sum_{n=0} a_n(x - x_0)$. It converges at $x_1 \in \mathbb{C}$ if $\sum_{n=0}^{\infty} a_n\,(x_1 - x_0)^n$ converges and diverges otherwise.

1.1.2 Power series as functions

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a (formal) power series (centred at 0). If $I \subseteq \mathbb{C}$ is a set such that for all $x_0 \in I$, $f(x_0)$ converges, we can define a function $F : I \to \mathbb{C}$ defined by $F(z) = f(z)$, where the right hand side means the limit of the series $\sum_{n=0}^{\infty} a_n z^n$.

1.1.3 Analytic Function

Let D be an open interval, and f a function defined on I. We say that f is analytic at $x_0 \in D$, if there is a formal power series g centred at c convergent on an interval $(x_0 - \delta, x_0 + \delta) \subseteq D$ for some $\delta > 0$, such that $f(x) = g(x)$ on $(c - \delta, c + \delta)$. We say f is analytic if that holds for every $c \in I$.

Example

The exponential function is analytic at 0. In fact, every power series with positive radius of convergence is analytic at its centre.

1.1.4 Coefficients of a formal power series

Let $f=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a formal power series centered at x_0 with radius of convergence $R>0$. Then f is smooth on (x_0-R, x_0+R), and $f^{(n)}$ is again a power series. Then in particular, $a_n=\frac{1}{n!} f^n(x_0)$.

1.1.5 Zeros of an Analytic Function

If z_0 is a regular point and not a singular point of an analytic function f and if $f\left(z_0\right)=0$, then z_0 is called a zero of f.

The point z_0 is called a zero of f of order m if in some neighbourhood of z_0, f can be expanded in a Taylor series of the form

$$f(z)=\sum^{\infty} a_n\left(z-z_0\right)^n, \text{ where } a_m \neq 0$$

1.1.6 Zeros of Order n

Let $f: D \rightarrow \mathbb{C}$ and If f is analytic at z_0, in a domain D, then we say that f has a zero of order $n \geq 1$ at z_0 if

$$0=f\left(z_0\right)=f^{\prime}\left(z_0\right)=\cdots=f^{(n-1)}\left(z_0\right)$$

and $f^{(n)}\left(z_0\right) \neq 0$. If $f^{(n)}\left(z_0\right)=0$ for all $n \geq 0$, then we call z_0 a zero of infinite order.

Example :

Let $f(z)=z^3-1$, then $f(z)$ has a zero of order 1 at $z_0=1$.

$$\begin{aligned}
f(1) &= 0 \\
f^{\prime}(z) &= 3 z^2 \\
f^{\prime}(1) &= 3
\end{aligned}$$

Therefore $f(z)$ has a zero of order 1 at $z_0=1$

1.1.7 Neighbourhood of a Zero of order m

A point $z=z_0$ is a zero of f of order m if and only if in some neighbourhood of z_0, f can be expressed in the form $f(z)=\left(z-z_0\right)^m g(z)$, where $g(z)$ is analytic at z_0 and $g\left(z_0\right) \neq 0$ and $f, g \in D \rightarrow \mathbb{C}$

Proof.

$\Longrightarrow$ Assume that z_0 is a zero of f of order m. Then there exists a neighbourhood of z_0 where we can expand f as

$$f(z)=\sum_{n=m}^{\infty} a_n\left(z-z_0\right)^n, \text{ where } a_m \neq 0$$

Then

$$\begin{aligned} f(z) &= (z-z_0)^m \sum_{n=m}^{\infty} a_n (z-z_0)^{n-m} \\ &= (z-z_0)^m \sum_{p=0}^{\infty} b_p (z-z_0)^p, \text{ where } n-m=p \text{ and } b_p = a_{p+m} \\ &= (z-z_0)^m g(z) \end{aligned}$$

where $g(z) = \sum_{p=0}^{\infty} b_p (z-z_0)^p$ is analytic at z_0 and $g(z_0) = b_0 = a_m \neq 0$.

$\Longleftarrow$

Now assume that in some neighbourhood of z_0, f can be expressed as

$$f(z) = (z-z_0)^m g(z)$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Then we can expand $g(z)$ in Taylor series about z_0 to obtain

$$g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \text{ where } a_0 = g(z_0) \neq 0$$

Therefore, in some neighbourhood of z_0, we have

$$\begin{aligned} f(z) = (z-z_0)^m \sum_{n=0}^{\infty} a_n (z-z_0)^n &= \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} \\ &= \sum_{p=m}^{\infty} b_p (z-z_0)^p, \text{ where } m+n=p \text{ and } b_p = a_{p-m} \end{aligned}$$

Since $b_m = a_0 \neq 0, z_0$ is a zero of f of order m. This proves the theorem.

1.1.8 Constant function

If f has a zero of infinite order at z_0. Then there is a $r > 0$ such that f is identically zero in $B_r(z_0)$ where $B_r(z_0) := \{z \in \mathbb{C} : |z-z_0| < r\}$

Proof

Now there exists a $r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{for all } z \in B_r(z_0)$$

by definition.

Now since z_0 is a zero of infinite order(using also the earlier lemma from subsection 1.1.3) then

$$a_n = \frac{f^{(n)}(z_0)}{n!} = 0 \qquad \text{for all } n \geq 0.$$

Hence $f(z) = 0$ for all $z \in B_r(z_0)$.

1.1.9 Extended Theorem

If f is analytic in a domain $D \subset \mathbb{C}$ and if f has a zero of infinite order in D, then f is identically zero. Earlier we proved that in every $B_r(z_0)$, $\exists r \in \mathbb{C}$ that f is identically zero but this can be extended to the whole domain.

1.1.10 Zeros are Isolated

Let $R \subset \mathbb{C}$ be some open set and let f be an analytic function defined on R. Then either f is a constant function, or the set $\{z \in R : f(z) = 0\}$ is totally disconnected ie all the zeros are isolated.

Proof

Suppose f has no zeroes in R. Then the set described in the theorem is the empty set, and we're done. So we suppose $\exists z_0 \in R$ such that $f(z_0) = 0$. Since f is analytic, there is a Taylor series for f at z_0 which converges for $|z - z_0| < R$. Now, since $f(z_0) = 0$, we know $a_0 = 0$. Other a_j may be 0 as well. So let k be the least number such that $a_j = 0$ for $0 \leq j < k$, and $a_k \neq 0$ Then we can write the Taylor series for f about z_0 as:

$$\sum_{n=k}^{\infty} a_n (z - z_0)^n = (z - z_0)^k \sum_{n=0}^{\infty} a_{n+k} (z - z_0)^n$$

where $a_k \neq 0$ (otherwise, we'd just start at $k+1$). Now we define a new function $g(z)$, as the sum on the right hand side, which is clearly analytic in $|z - z_0| < R$. Since it is analytic here, it is also continuous here. Since $g(z_0) = a_k \neq 0, \exists \varepsilon > 0$ so that $\forall z$ such that $|z - z_0| < \varepsilon, |g(z) - a_k| < \frac{|a_k|}{2}$. But then $g(z)$ cannot possibly be 0 in that disk. Hence the result.

1.1.11 Open Disk

The open disk of radius r around z_0 is the set of points z with $|z - z_0| < r$, i.e. all points within distance r of z_0

1.1.12 Closed Disk

The open disk of radius r around z_0 is the set of points z with $|z - z_0| \leq r$, i.e. all points within distance r of z_0

1.1.13 Open Deleted Disk

The open deleted disk of radius r around z_0 is the set of points z with $0 < |z - z_0| < r$. That is, we remove the center z_0 from the open disk. A deleted disk is also called a punctured disk.

1.1.14 Deleted Neighbourhood

A deleted neighbourhood of a point p is a neighbourhood of p, without $\{p\}$.

Example

The interval $(-1,1) = \{y : -1 < y < 1\}$ is a neighbourhood of $p = 0$ in the real line, so the set $(-1,0) \cup (0,1) = (-1,1) \setminus \{0\}$ is a deleted neighbourhood of 0.

1.1.15 Isolated

The singleton set x is an open set in the topological space $S \subseteq X$. If the space X is a Euclidean space, then x is an isolated point of S if there exists an open ball around x which contains no other points of S.

Example

For the set $S = \{0\} \cup [1,2]$, the point 0 is an isolated point.

1.1.16 Pointwise Convergence

Let $\neq D \subset \mathbb{C}^N$, and let $f, f_1, f_2, \ldots$ be $\mathbb{C}$ -valued functions on D. Then the sequence $(f_n)_{n=1}^{\infty}$ is said to converge pointwise to f on D if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

holds for each $x \in D$. *Runde* (2021) Similarly can also be defined in $\mathbb{R}^N$ and also the following subsequent theorems.

Examples

For $n \in \mathbb{N}$, let

$$f_n : [0,1] \to \mathbb{R}, \quad x \mapsto x^n$$

so that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

Let

$$f : [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

It follows that $f_n \to f$ pointwise on [0,1] .

1.1.17 Uniform Convergence

Let $\neq D \subset \mathbb{C}^N$, and let $f, f_1, f_2, \ldots$ be $\mathbb{C}$ -valued functions on D. Then the sequence $(f_n)_{n=1}^{\infty}$ is said to converge uniformly to f on D if, for each $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq n_\varepsilon$ and for all $x \in D$.

Remark

Let us introduce the uniform norm

$$\|g\|_D = \sup_{z \in D} |g(z)| \quad \text{for} \quad g : D \to \mathbb{C}.$$

Then $f_n \to f$ uniformly in D if and only if $\|f_n - f\|_D \to 0$ as $n \to \infty$. We wil omit the use of this norm and stick to the usual norm as above.

Examples

For $n \in \mathbb{N}$, let

$$f_n : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{\sin(n\pi x)}{n}$$

Since

$$\left| \frac{\sin(n\pi x)}{n} \right| \leq \frac{1}{n}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, it follows that $f_n \to 0$ uniformly on $\mathbb{R}$.

1.1.18 Uniform Limit of Continuous Functions

Let $\neq D \subset \mathbb{C}^N$, and let $f, f_1, f_2, \ldots$ be functions on D such that $f_n \to f$ uniformly on D and such that $f_1, f_2, \ldots$ are continuous. Then f is continuous.

Proof

Let $\varepsilon > 0$, and let $x_0 \in D$. Choose $n_\varepsilon \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$$

for all $n \geq n_\varepsilon$ and for all $x \in D$. Since f_{n_e} is continuous, there is $\delta > 0$ such that $|f_{n_\varepsilon}(x) - f_{n_\varepsilon}(x_0)| < \frac{\varepsilon}{3}$ for all $x \in D$ with $\|x - x_0\| < \delta$. Fox any such x we obtain:

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_{n_\varepsilon}(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_{n_\varepsilon}(x) - f_{n_\varepsilon}(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_{n_\varepsilon}(x_0) - f(x_0)|}_{<\frac{\varepsilon}{3}} < \varepsilon.$$

Hence, f is continuous at x_0. Since $x_0 \in D$ was arbitrary, f is continuos on all of D.

1.1.19 Uniform Cauchy Sequence

Let $\neq D \subset \mathbb{C}^N$. A sequence $(f_n)_{n=1}^\infty$ of $\mathbb{C}$ -valued functions on D is called a uniform Cauchy sequence on D if, for each $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $x \in D$ and all $n, m \geq n_\varepsilon$

1.1.20 Weierstrass M-test

Let $\neq D \subset \mathbb{C}^N$, let $(f_n)_{n=1}^\infty$ be a sequence of $\mathbb{C}$-valued functions on D and suppose that, for each $n \in \mathbb{N}$, there is $M_n \geq 0$ such that $|f_n(x)| \leq M_n$ for $x \in D$ and such that $\sum_{n=1}^\infty M_n < \infty$. Then $\sum_{n=1}^\infty f_n$ converges uniformly and absolutely on D.

Proof of Weierstrass M-test

Let $\varepsilon > 0$ and choose $n_{\tilde{\varepsilon}} \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} M_k < \varepsilon$$

for all $n \geq m \geq n_\varepsilon$. For all such n and m and for all $x \in D$, we obtain that

$$\left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon$$

Hence, the sequence $(\sum_{k=1}^{n} f_k)_{n=1}^{\infty}$ is uniformly Cauchy on D and thus uniformly convergent. It is easy to see that the convergence is even absolute.

1.1.21 Differential Operator

Let $R \subset \mathbb{C}$ be an interval(open and connected set) and n, k be positive integers.

Consider the map

$$D : C^1(R) \to C(R)$$

given by $D(f) = f'$. More generally, for any $k \in \{1, \ldots, n\}$, consider the map

$$D^k : C^k(R) \to C(R)$$

given by $D^k(f) = f^{(k)}$, where $f^{(k)}$ denotes the k -th derivative of f. Observe that $D^k = D \circ D \circ \cdots \circ D(k \text{ times })$. By convention, $D^0 = Id$ (the identity map). The operators (or maps) D^k are called differentiation operators.

Definition

A differential operator from $C^n(R)$ to $C(R)$ is a map

$$L : C^n(R) \to C(R)$$

which can be expressed as a function of the differentiation operator D.

Examples

Let $L = D^n$ or $L = e^D$

Properties

- $L : C^n(R) \to C(R)$ is said to be linear if for any $y(x), y_1(x), y_2(x) \in C^n(R)$ and $c \in \mathbb{R}$

$$L(y_1 + y_2) = L(y_1) + L(y_2) \text{ and } L(cy) = cL(y)$$

Linear ODE

An ODE given by $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$ on an interval R is said to be linear if it can be written as $L(y)(x) = g(x)$, where $L : C^n(R) \to C(R)$ is a linear differential operator.

1.1.22 Homogeneous Linear n'th order ODE

Suppose that $a_j(z) \in C(R)$ and $a_n(z) = 1$ for all $z \in R$. Let $z_0 \in R$. Then the initial value problem (IVP)

$$(Ly)(z) = 0, \quad y^{(j)}(z_0) = y_j, j = 0, \ldots, n-1$$

where $y_j \in R$ and $L(y)(z) := y^{(n)}(z) + a_{n-1}(z)y^{(n-1)}(z) + \cdots + a_1(z)y'(z) + a_0(z)y(z)$ has a unique solution $y(z)$ for all $z \in R$.

Superposition Principle

Let $y_i \in C^n(R), i = 1, \cdots, n$ be any solutions of $L(y)(z) = 0$ on I. Then $y(z) = c_1y_1(z) + c_2y_2(z) + \cdots + c_ny_n(z)$, where $c_i, i = 1, \cdots, n$ are arbitrary constants, is also a solution on R

Kernel

Consider the linear differential operator L where

$$L(y) := a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y$$

where $a_i : R \to \mathbb{C}$ are given functions. Given $g(z) \in C(R)$, find $y \in C^n(R)$ such that $L(y) = g(z)$. Since $L : C^n(R) \to C(R)$ is a linear transformation, the solution set of

$$L(y) = g(z) + y_p$$

is given by

$$\text{Ker}(L)$$

where y_p is a particular solution (PS) satisfying $L(y_P) = g$ and $\text{Ker}(L) = \{y \in C^n(R) \mid L(y) = 0\}$

1.1.23 Gronwalls lemma

Let $u(z)$ and $h(z) \geq 0$ be continuous in $[a, b] \subset \mathbb{R}$ such that

$$u(z) \leq C + \int_a^z u(s)h(s)ds - \mathbf{1}$$

for some constant C and for all $a \leq z \leq b$. Then

$$u(z) \leq Ce^{\int_a^z h(s)ds}$$

for all $a \leq z \leq b$. To see this, differentiate both sides of **1** and use the second fundamental theorem of calculus to obtain

$$u'(z) - u(z)h(z) \leq 0$$

Multiplying both sides by the integrating factor $e^{-\int_a^z h(s)ds}$ to obtain

$$\frac{d}{dz}\left[e^{-\int_a^z h(s)ds}u(z)\right] \leq 0$$

Integrating both sides from a to z, we find

$$e^{-\int_a^z h(s)ds}u(z) - u(a) \leq 0$$

Hence proved.

1.1.24 Closed Bounded Set

Let $D \subset \mathbb{C}$ be a closed, bounded set and let $f(z)$ be a continuous complex function in D then $f(z)$ is bounded in D.

Assume $f(z)$ is not bounded on D. Then $\forall n \in \mathbb{N}, \exists z_n \in D$ s.t. $|f(z_n)| > n$. Construct the sequence $(z_n)_{n=1}^{\infty} \subset E$ from these z_n. Note that $(z_n)_{n=1}^{\infty}$ is bounded, as D is bounded. Then by Bolzano-Weierstrass, $(z_n)_{n=1}^{\infty}$ has a limit point L, and so there exists a subsequence $\left(z_{n_k}\right)_{k=1}^{\infty}$ which converges to L. Moreover, $L \in D$ since D is a closed set. This implies $\lim_{k\to\infty} f\left(z_{n_k}\right) = f(L)$, and so $\lim_{k\to\infty}\left|f\left(z_{n_k}\right)\right| = |f(L)|$ because f is continuous on D, and $f(z)$ continuous implies $|f(z)|$ is continuous.

1.2 Linear Algebra Background

All of the definitions/theorems and proofs are from the standard 227/127 textbook and have been followed similarly. *Kuttler* (2019)

1.2.1 Vector Spaces

Let $\mathbb{F}$ be a field. An $\mathbb{F}$ -véctor space or simply vector space if $\mathbb{F}$ is understood is a triple $(V, +, \cdot)$ where V is a nonempty set and $+$ is an associative operation on V, called the addition of V, and . is a map $\mathbb{F} \times V \to V$ called the scalar multiplication (which associates to each $c \in \mathbb{F}$ and each $v \in V$ an element $cv = c \cdot v \in V$), such that the following properties hold:

- The addition is commutative: $v + w = w + v$ for all $v, w \in V$.
- There is an identity element for the addition: There is an element 0 called the zero vector or simply zero of V such that $0 + v = v + 0 = v$ for all $v \in V$.
- Each element of V has an additive inverse: for each $v \in V$ there is an element $-v$ of V such that $v + (-v) = 0$
- The scalar multiplication is associative: for each $a, b \in \mathbb{F}$ and each $v \in V$, we have $a(bv) = (ab)v$
- $1 \in \mathbb{F}$ is an identity element for the scalar multiplication: $1v = v$ for all $v \in V$.
- The scalar multiplication is distributive in the following two senses: for each $a, b \in \mathbb{F}$ and each $v \in V$ we have $(a+b) \cdot v = av + bv$; and for each $c \in \mathbb{F}$ and $v, w \in V$ also $c \cdot (v + w) = cv + cw$.

Examples

- A vector space over the field $\mathbb{R}$ of real numbers is often called Real, and a vector space over the field $\mathbb{C}$ of complex numbers is often called Complex.

1.2.2 Subspaces

Let V be an $\mathbb{F}$ -vector space. A subset $W \subseteq V$ is called a subspace of V if it satisfies the following three properties:

- W is not empty.
- If $v, w \in W$ then also $v + w \in W$.
- If $w \in W$ and $r \in \mathbb{F}$ then $rv \in W$.

Examples

A function $f : \mathbb{R} \to \mathbb{R}$ is called a polynomial function if there exists a (fixed) list of real numbers $a_0, a_1, \dots, a_n$ such that for each $x \in \mathbb{R}$

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Let $\mathcal{P}(\mathbb{R})$ be the set of all polynomial functions on $\mathbb{R}$. Then $\mathcal{P}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R})$ is a subspace.

1.2.3 Linear Independence/Dependence

An ordered list $(v_1, v_2, \dots, v_p)$ of vectors $v_1, v_2, \dots, v_p \in V$ is called linearly dependent if there are scalars $c_1, c_2, \dots, c_p \in \mathbb{F}$ not all zero such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

Such a formula is called a Linear Dependence relation. We also call the vectors $v_1, v_2, \dots, v_p$ linearly dependent if the list $(v_1, v_2, \dots, v_p)$ is.

The list $(v_1, v_2, \dots, v_p)$ is called linearly independent if it is not linearly dependent. In other words, it is linearly independent if

$$c_1 = c_2 = \cdots = c_p = 0.$$

Thus, $(v_1, v_2, \dots, v_p)$ is linearly independent if and only if there is one and only one way to write 0 as a linear combination of the $v_i : 0 = 0v_1 + 0v_2 + \cdots + 0v_p$. If this is the case we also say the vectors $v_1, v_2, \dots, v_p$ are linearly independent.

Examples

- In $\mathbb{F}^n$, the vectors $e_1, e_2, \dots, e_n$ are linearly independent. Indeed, suppose $c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = 0$. Then observe that

$$c_1 e_1 + \cdots + c_n e_n = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

  if and only if all $c_i = 0$.

- In $\mathbb{R}^3$, the three vectors

$$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

  are linearly independent.

1.2.4 Span

Let $v_1, v_2, \ldots, v_n \in V (n > 0)$. Then $\text{Span}(v_1, \ldots, v_n)$ is a subspace of V. In fact it is the minimal subspace containing $v_1, v_2, \ldots, v_n$ in the following senise: if W is any subspace of V containing $v_1, v_2, \ldots, v_n$ as elements, then $\text{Span}(v_1, v_2, \ldots, v_n) \subseteq W$. Thus,

$$\text{Span}(v_1, v_2, \ldots, v_n) = \bigcap_{\substack{W \subseteq V \\ v_1, v_2, \ldots, v_n \in W}} W$$

where the intersection ranges over all subspace of V that contain $v_1, v_2, \ldots, v_n$.

Examples

If $A_1, A_2, \ldots, A_n \in \mathbb{F}^m$ are the columns of the matrix $A \in M_{m \times n}(\mathbb{F})$, then we call

$$\text{Col}(A) = \text{Span}(A_1, A_2, \ldots, A_n)$$

the column space of A. It is a subspace of $M_{m \times 1}(\mathbb{F})$ which as usual we identify with $\mathbb{F}^m$ It is the set of all $B \in \mathbb{F}^m$ for which the matrix equation $AX = B$ has a solution: indeed, $AX = B$ has a solution if and only if B can be expressed as a linear combination of the columns $A_1, A_2, \ldots, A_n$ of A.

1.2.5 Basis

Let V be a vector space. A basis is a linearly independent ordered list of generators. Thus, $\mathcal{B} \subseteq V$ is a basis if and only if $\mathcal{B}$ is linearly independent and $\text{Span}(\mathcal{B}) = V$. We write

$$\mathcal{B} = (v_1, v_2, \ldots, v_n)$$

if $v_1, v_2, \ldots, v_n$ are the elements of $\mathcal{B}$ (in order). By convention, the empty set is a basis for $V = \{0\}$.

1.2.6 Examples

Suppose $V = \mathbb{F}^n$ .Then $\mathcal{E} = (e_1, e_2, \ldots, e_n)$ is a basis. (Both $\mathcal{E}$ is linearly independent and that $\text{Span}(\mathcal{E}) = \mathbb{F}^n$.) For $v \in \mathbb{F}^n$ we have $\mathcal{E}v = v$, so $[v]_{\mathcal{E}} = v$. This makes this particular basis a little special; it is therefore often referred to as the standard basis of $\mathbb{F}^n$.

1.2.7 Exchange Lemam

Let V be a vector space spanned by elements $v_1, v_2, \ldots, v_n$, say. Lel $v = c_1 v_1 + \cdots + c_n v_n \in V$ be a vector. If $c_i \neq 0$, then

$$V = \operatorname{Span}(v_1, v_2, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n)$$

1.2.8 Theorem for Independence

Let V be a vector space generated by finitely many elements $(v_1, v_2, \ldots, v_n)$, say. If $(w_1, w_2, \ldots, w_k)$ is a linearly independent list of elements of V, then $k \leq n$.

Proof

Let $L = (v_1, v_2, \ldots, v_n)$ and $M = (w_1, w_2, \ldots, w_k)$. If $n = 0$ (that is, if L is empty), then $V = \{0\}$, so any number of elements of V are linearly dependent. Hence $k = 0$ as well. We may therefore assume that $n > 0$. Suppose precisely $m \geq 0$ of the elements of M are also elements of L. By reordering if necessary, we may assume that $w_1 = v_1, w_2 = v_2, \ldots, w_m = v_m$. We will now show how to increase m by 1 if $k - m > 0$. In this case, $w_{m+1} \notin L$. We may write $w_{m+1} = c_1 v_1 + \cdots + c_m v_m$ for suitable $c_i \in \mathbb{F}$.

Claim: At least one c_i with $i > m$ must be nonzero. Indeed, otherwise $c_{m+1} = c_{m+2} = \cdots = c_n = 0$ and

$$w_{m+1} = c_1 v_1 + \cdots + c_m v_m = c_1 w_1 + \cdots + c_m w_m$$

contradicting the fact that M is linearly independent. This proves the claim. So pick one such i (ie. $i > m$ and $c_i \neq 0$). By the Exchange Lemma, we can replace v_i by w_i in L, obtaining a new list of generators L' which has $m + 1$ elements in common with M and still satisfies $V = \operatorname{Span}(L')$ This process can be repeated as long as $k - m > 0$. Thus eventually, all elements of M must be elements of the newly created list L'. In particular, $n \geq k$.

1.2.9 Dimension

Let V be a vector space with basis $\mathcal{B} = (v_1, v_2, \ldots, v_n)$. The uniquely determined integer n is called the dimension of V and denoted $\dim V$.

The empty set by convention is a basis for $V = \{0\}$ (it is after all a linearly independent set that spans V). So $\dim\{0\} = 0$. If V does not have a (finite) basis, then we say $\dim V = \infty$.

Example

As expected $\dim \mathbb{R} = 1$ (the list with one element $(1_{\mathbb{R}})$ is a basis), $\dim \mathbb{R}^2 = 2$ and $\dim \mathbb{R}^3 = 3$. More generally, (3.23)

$$\dim \mathbb{F}^n = n$$

- The standard basis, $\mathcal{E} = (e_1, e_2, \ldots, e_n)$ of $\mathbb{F}^n$ has exactly n elements.

- $\dim M_{m\times n}(\mathbb{F}) = mn$. Here we may choose as a basis a list whose elements are precisely the mn matrix units e_{ij} (in any ordering).

Chapter 2

Supplementary Lemmas'

2.1 Lemma 1

If $f(x)$ is a solution of

$$y^{(n)}(z) = a_{n-1}(z)y^{(n-1)}(z) + \cdots + a_0(z)y(z)$$

then $f := 0, \ \forall z \in R \subset \mathbb{C}$.

2.1.1 Proof

Before going over the proof I would first like to present an example before proving the general case.

Let us saying we have the following equation when $n = 1$

$$y^1 = a_0(z)y, \ a_0 = g(z)$$

Now this is easily solvable and the general solution is given by

$$y = Ce^{\int a_0(z)dz}, \forall C \in R$$

Now note that the exponential function has no zeros $\iff$ no poles hence there does not

$$\exists z_0, \text{ such that } e^{z_0} = 0$$

Therefore

$$\nexists z_0 \text{ s.t } e^{\{\int a_0(z)dz = h(z_0)\}} = 0 \text{ unless } \int a_0(z)dz = \ln g(z) \implies e^{\ln g(z)} = g(z) \ \exists z_0 \text{ s.t } g(z_0) = 0$$

Now considering the general solution let us divide this problem into two sub parts

- Now consider a function $a_0(z) = \frac{g'(z)}{g(z)}$ then $e^{\int a_0(z)dz} = \ln g(z)$. Then the general solution is $y = Cg(z)$ and $y' = Cg'(z)$ and let us assume that $C \neq 0$. Then by assumption $\exists z_0, z_1$ s.t $g(z_0) = 0 = g'(z_1)$, but clearly if this is the case then the function $a_0(z)$ is not analytic in the region R. If we exclude the point's at which the function $a_0(z)$ is 0 then we get the region R at which the function has no zeros and hence our general solution can never be 0 and hence has no zero of order 2. Therefore the only way for $y = Cg(z) = 0$ is for $C = 0$ and hence $y = 0$ which is a zero of order infinite order.

- Considering any other general function yields the same answer as before as $\nexists z_0$ s.t $e^{\{\int a_0(z)dx=h(z_0)\}}=0$ and hence the only way for $y=Ce^{\int a_0(z)dz}=0$ is $y=0$.

Hence the only solution for this example to have a zero of order 2 (or a zero of infinite order which proved by 1.18 and 1.19 could be extended to the whole of R) would be $f:=0$.

Now let us prove the general case. From the hypothesis of our lemma and after substituting $f(z)$(which is a solution) in our original equation ie

$$y^{(n)}(z)=a_{n-1}(z)y^{(n-1)}(z)+\cdots+a_0(z)y(z)$$

it is clear that f has a zero of order atleast $n+1$. $f(z_0)=0,\ z_0\in R$. Now we prove this by contradiction. Suppose that f is not identically 0 ie $f:\not\equiv 0$ then $\exists p\geq 1$ such that f has a zero of order $n+p$ at z_0. Then by 1.1.7 we have

$$f(z)=(z-z_0)^{n+p}\cdot g(z)$$

and we already know that $g(z_0)\neq 0$ and $g(z)$ is analytic in the Neighbourhood of z_0. To make simplifications easier let $k=n+p$ and then let

$$f(z)=(z-z_0)^k\cdot g(z)$$

Now taking derivatives of f we find that

$$f'=k\cdot(z-z_0)^{k-1}\cdot g(z)+g'(z)\cdot(z-z_0)^k$$

$$f''=g''(z)\cdot(z-z_0)^k+k\cdot(z-z_0)^{k-1}\cdot g'(z)+k(k-1)\cdot(z-z_0)^{k-2}\cdot g(z)+g'(z)\cdot k\cdot(z-z_0)^{k-1}$$

Similarly we can find all derivatives upto f^n ie the n'th derivative and substitute these all in the equation

$$y^{(n)}(z)=a_{n-1}(z)y^{(n-1)}(z)+\cdots+a_0(z)y(z)$$

Example when $n=1$
Then substituting in the original equation above we gain.

$$k\cdot(z-z_0)^{k-1}\cdot g(z)+g'(z)\cdot(z-z_0)^k=a_0\cdot((z-z_0)^k\cdot g(z))$$

$$k\cdot(z-z_0)^{k-1}\cdot g(z)-a_0\cdot((z-z_0)^k\cdot g(z))=-g'(z)\cdot(z-z_0)^k$$

$$k\cdot g(z)(z-z_0)^{k-1}(1-a_0(z-z_0))=(z-z_0)^k\cdot g'(z)$$

$$k\cdot g(z)(z-z_0)^{k-1}=(z-z_0)^k\cdot\frac{-g'(z)}{(1-a_0(z-z_0))}$$

Now let $p(z)=\frac{-g'(z)}{(1-a_0(z-z_0))}$ and this implies

$$k \cdot g(z)(z-z_0)^{k-1} = (z-z_0)^k \cdot p(z)$$

Similarly grouping only the $g(z)$ coefficients together and other terms naming it as function $p(z)$ we have

$$k(k-1)\cdots(k-n+1)g(z)(z-z_0)^{k-n} = (z-z_0)^{k-n+1}p(z)$$

$$k(k-1)\cdots(k-n+1)g(z) = (z-z_0)p(z)$$

Now $p(z)$ is clearly continuous as it is just gonna be a bunch of coefficients and derivatives of $g(z)$ and hence analytic too. Now the above equation only holds true in some deleted neighbourhood of z_0 as if it was true including z_0 then we couldn't really define $f(z) = (z-z_0)^k g(z)$ as $f(z_0) = 0$.

Now finally

$$\lim_{z \to z_0} k(k-1)\cdots(k-n+1) \cdot g(z_0) = \lim_{z \to z_0}(z-z_0)p(z)$$

$$\lim_{z \to z_0} k(k-1)\cdots(k-n+1) \cdot g(z_0) = 0$$

And finally we know that

$$k(k-1)....(k-n+1) \neq 0$$

Hence the only possibility is $g(z_0) = 0$

which is a contradiction to our statement and 1.1.7. Hence

$$f := 0$$

2.2 Solution Space

Now before proving the next lemma let us first understand what does a solution space mean. *Krom* (1979)

2.2.1 Definition

The solution space of a linear homogeneous differential equation is a vector space over any field F. This is denoted by $V_{\mathbb{F}}^n$ and the dimension of it denoted by $\dim(V_F^n)$.
Let $R \subset \mathbb{C}$ Then V_R^n is a linear space of analytic functions over the field of Complex numbers ($\mathbb{C}$).

2.2.2 Example

Let F be the vector space with the basis $\{t, e^t\}$. We expand the determinant

$$\begin{vmatrix} y & t & e^t \\ y' & 1 & e^t \\ y'' & 0 & e^t \end{vmatrix}$$

by the elements of the first column to get $(t-1)y''-ty'+y=0$.

An important example is the constant coefficient differential equation

$$a_n\frac{d^ny}{dz^n}+\cdots a_1\frac{dy}{dz}+a_0y=0,\ \text{with}\ a_n\neq 0$$

A basis for the solution space F is given by

$$\left\{z^ke^{z\lambda i}\right\}k=0,1,\ldots,m_i-1;i=1,\ldots,s$$

where $\lambda_1,\ldots,\lambda_s$ are the distinct roots of the characteristic equation

$$f(\lambda)=a_n\lambda^n+\cdots+a_1\lambda+a_0=0$$

and λ_i has multiplicity m_1.

2.3 Lemma 2

$$\dim(V_R^n)\leq n$$

2.3.1 Proof

Now this should be obvious due to the fact that we proved this in 1.2.8 ie the Theorem for independence due to the fact that $(y_1,y_2\cdots,y_n)$ generate the solution space and any other list of such vectors of dimension k will always be less than or equal to n. The next proof follows by the way the paper describes it and goes as follows.

Let us assume that $\dim(V_R^n)>n$ and obtain a contradiction. Let $(y_1,y_2,\cdots,y_{n+1})$ be a linearly independent list of our solution space (V_R^n).

Consider the system of n linear equations with $n+1$ unknowns and $z\in R$ as follows

$$\sum_{k=1}^{n+1}x_k\cdot y_k^i(z)=0,\ i\in\{0,1,2\cdots,n-1\}$$

This system looks like

$$x_1\cdot y_1^0(z)+x_2\cdot y_2^0(z)\cdots+x_{n+1}\cdot y_{n+1}^0(z)=0$$
$$x_1\cdot y_1^1(z)+x_2\cdot y_2^1(z)\cdots+x_{n+1}\cdot y_{n+1}^1(z)=0$$
$$\cdots$$
$$\cdots$$
$$x_1\cdot y_1^{n-1}(z)+x_2\cdot y_2^{n-1}(z)\cdots+x_{n+1}\cdot y_{n+1}^{n-1}(z)=0$$

Now this system has a non trivial solution say $(s_1,s_2,\cdots,s_{n+1})$ and this implies that the solution

$$\sum_{k=1}^{n+1}s_i\cdot y_i$$

satisfies

$$y^{(n)}(z) = a_{n-1}(z)y^{(n-1)}(z) + \cdots + a_0(z)y(z)$$

then this solution has a zero of order n at z and by Lemma 1 this implies that this solution is identically 0 ie

$$\sum_{k=1}^{n+1} s_i \cdot y_i := 0 \in R$$

But clearly this is a contradiction since we assumed that we have a non trivial solution and that $(y_1, y_2, \cdots, y_{n+1})$ are linearly independent in R.

Clearly this is directly correlated to the previous proof of the Independence Theorem.

Chapter 3

Proof of the Main Theorem

3.1 Proof of the Main Theorem

Finally we have reached the gist of the paper and ready to prove the Main theorem after all the prerequisites have been met. Any additional Lemma/Theorems that are required have been proved subsequently. *Bose* (1982)

3.1.1 Statement

For any arbitrary region $R \subset \mathbb{C}$, The solution space of the homogeneous linear differential equation of order n and where every coefficient $a_j(z), j = 0, 1, 2, \ldots, n-1$, is continuous

$$y^{(n)}(z) + a_{n-1}(z)y^{(n-1)}(z) + \cdots + a_0(z)y(z) = 0$$

is n -dimensional ($\dim(V_R^n) = n$) if and only if every coefficient $a_j(z), j = 0, 1, 2, \ldots, n-1$ are analytic.

3.1.2 Proof

$\Longleftarrow$ We need to first prove that if all the coefficient's $a_j(z), j = 0, 1, 2, \ldots, n-1$, are analytic in $R \subset \mathbb{C}$ which are all also continuous then $\dim(V_R^n) = n$.

Now this is basically to prove Theorem 1.2.2 ie Suppose that $a_j(z) \in C(R)$ and $a_n(z) = 1$ for all $z \in R$. Let $z_0 \in R$. Then the initial value problem (**Eqn 1**)

$$(Ly)(z) = 0, \quad y^{(j)}(z_0) = y_j, j = 0, \ldots, n-1 \;\; z_0 \in R$$

where $y_j \in R$ and $L(y)(z) := y^{(n)}(z) + a_{n-1}(z)y^{(n-1)}(z) + \cdots + a_1(z)y'(z) + a_0(z)y(z)$ has a unique solution $y(z)$ in a closed bounded set $E \subset R$ that contains z_0.

3.1.3 Existence and Uniqueness Theorem

Existence

The existence of a local solution is obtained here by transforming the problem into a first order system. This is done by introducing the variables(similar as to the case we did in the notes)

$$x_1 = y, x_2 = y', \cdots, x_n = y^{(n-1)}$$

In this case, we have

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ \vdots &= \vdots \\ x_{n-1}' &= x_n \\ x_n' &= -a_{n-1}(z)x_n - \cdots - a_1(z)x_2 - a_0(z)x_1 \end{aligned}$$

Thus, we can write the initial-value problem as a system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

or in a more compact form

$$\mathbf{x}'(z) = A(z)\mathbf{x}(z) + \mathbf{b}(z), \quad \mathbf{x}(z_0) = \mathbf{x}_0$$

$$\text{and where } \mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}$$

$$\mathbf{x}(z) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b}(z) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Therefore since $b(z)$ is a 0 vector hence we can omit this out of our equation and hence our compact form equation(**Eqn 2**) becomes

$$\mathbf{x}'(z) = A(z)\mathbf{x}(z), \quad \mathbf{x}(z_0) = \mathbf{x}_0$$

Note that if $y(z)$ is a solution of **Eqn 1** then the vector-valued function

$$\mathbf{x}(z) = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

is a solution to **Eqn 2**. Conversely, if the vector

$$\mathbf{x}(z) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

is a solution of **Eqn 2** then $x_1' = x_2, x_1'' = x_3, \cdots, x_1^{(n-1)} = x_n$.

Hence

$$x_1^{(n)} = x_n' = -a_{n-1}(z)x_n - a_{n-2}(z)x_{n-1} - \cdots - a_0(z)x_1$$

and

$$x_1^{(n)} + a_{n-1}(z)x_1^{(n-1)} + a_{n-2}(z)x_1^{(n-2)} + \cdots + a_0(z)x_1 = 0$$

or

$$y^{(n)} + a_{n-1}(z)y^{(n-1)} + a_{n-2}(z)y^{(n-2)} + \cdots + a_0(z)y = 0$$

which means that $y = x_1(z)$ is a solution to **Eqn 1**.

Moreover, $x_1(z_0) = y_0, x_1'(z_0) = x_2(z_0) = y_1, \cdots, x_1^{(n-1)}(z_0) = x_n(z_0) = y_{n-1}$. That is, $x_1(z)$ satisfies the initial conditions of **Eqn 1**.

Next, we start by reformulating **Eqn 2** as an equivalent integral equation. Integration of both sides of **Eqn 2** yields (**Eqn 3**)

$$\int_{z_0}^{z} \mathbf{x}'(s)ds = \int_{z_0}^{z} [\mathbf{A}(s)\mathbf{x}(s)]ds$$

Applying the Fundamental Theorem of Calculus to the left side of **Eqn 3** yields

$$\mathbf{x}(\mathbf{z}) = \mathbf{x}(z_0) + \int_{z_0}^{z} [\mathbf{A}(s)\mathbf{x}(s)]ds, \ \mathbf{x}(z_0) = \mathbf{x}_0 - \textbf{Eqn 4}$$

Thus, a solution of **Eqn 4** is also a solution to **Eqn 2** and vice versa. Now To prove the existence of a solution, we shall use the method of successive approximation.

Letting

$$\mathbf{x}_0 = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

we can introduce Picard's iterations defined recursively as follows:

$$\begin{aligned}
&\mathbf{x}_0(z) = \mathbf{x}_0 \\
&\mathbf{x}_1(z) = \mathbf{x}_0 + \textstyle\int_{z_0}^{z} [\mathbf{A}(s)\mathbf{x}_0(s)]\, ds \\
&\mathbf{x}_2(z) = \mathbf{x}_0 + \textstyle\int_{z_0}^{z} [\mathbf{A}(s)\mathbf{x}_1(s)]\, ds \\
&\vdots \\
&\mathbf{x}_N(z) = \mathbf{x}_0 + \textstyle\int_{z_0}^{z} [\mathbf{A}(s)\mathbf{x}_{N-1}(s)]\, ds
\end{aligned}$$

Let

$$\mathbf{x}_N(z) = \begin{bmatrix} x_{1,N} \\ x_{2,N} \\ \vdots \\ x_{n,N} \end{bmatrix}$$

For $i = 1, 2, \cdots, n$, we are going to show that the sequence $\{x_{i,N}(z)\}_{N=1}^{\infty}$ converges uniformly to a function $x_i(z)$ such that $\mathbf{x}(t)$ (with components $x_1, x_2, \cdots, x_n$) is a solution to **Eqn 4** and hence a solution to **Eqn 2**.

Let E be a closed bounded set containing z_0 and contained in $R \subset \mathbb{C}$. For $i = 0, 1, \cdots, n-1$, the function $a_i(z)$ is continuous in $z \in R$ and in particular it is continuous in $E \subseteq R$. We know from analysis then that a continuous function on a closed bounded set is bounded(Theorem 1.1.24). Hence, there exist positive constants $k_0, k_1, \cdots, k_{n-1}$ such that

$$\max_{z \in E} |a_0(z)| \leq k_0, \quad \max_{z \in E} |a_1(z)| \leq k_1, \cdots, \max_{z \in E} |a_{n-1}(z)| \leq k_{n-1}$$

This implies that

$$\begin{aligned} \|\mathbf{A}(z)\mathbf{x}(z)\| &= |x_2| + |x_3| + \cdots + |x_{n-1}| + |a_0 x_1 + a_1 x_2 + \cdots + a_{n-1} x_n| \\ &\leq |x_2| + |x_3| + \cdots + |x_{n-1}| + |a_0|\,|x_1| + |a_1|\,|x_2| + \cdots + |a_{n-1}|\,|x_n| \\ &\leq k_0\,|x_1| + (1 + k_1)\,|x_2| + \cdots + (1 + k_{n-2})\,|x_{n-1}| + k_{n-1}\,|x_n| \\ &\leq K \cdot \|\mathbf{x}\| \end{aligned}$$

for all $z \in E$, where we define

$$||\mathbf{x}|| = |x_1| + |x_2| + \cdots + |x_n|$$

and where

$$K = k_0 + (1 + k_1) + \cdots + (1 + k_{n-2}) + k_{n-1}$$

For $i = 1, 2, \cdots, n$, we have

$$\begin{aligned} |x_{i,N} - x_{i,N-1}| \leq \|\mathbf{x}_N - \mathbf{x}_{N-1}\| &\leq \int_{z_0}^{z} \|\mathbf{A}(s) \cdot (\mathbf{x}_{N-1} - \mathbf{x}_{N-2})\|\, ds \\ &\leq K \int_{z_0}^{z} \|\mathbf{x}_{N-1} - \mathbf{x}_{N-2}\|\, ds \end{aligned}$$

Also

$$\begin{aligned} \|\mathbf{x}_1 - \mathbf{x}_0\| &\leq \int_{z_0}^{z} \|[\mathbf{A}(s) \cdot \mathbf{x}_0]\|\, ds \\ &\leq M\,(z - z_0) \end{aligned}$$

where

$$M = K\,\|\mathbf{x}_0\|$$

Induction on $N \geq 1$ yields

$$\|\mathbf{x}_N - \mathbf{x}_{N-1}\| \leq M K^{N-1} \frac{(z - z_0)^N}{N!}$$

By our assumption that R is an open connected set then the set $R = \{(x+y\cdot i) \in \mathbb{C} : x \in (e,f),\ y \in (c,d)\}$ can be represented this way and hence let $b = (f-e)$ and $a = i\cdot(d-c)$
Since $N! \geq (N-1)!$ and $z - z_0 < b - a$ we have

$$\|\mathbf{x}_N - \mathbf{x}_{N-1}\| \leq MK^{N-1}\frac{(z-z_0)^N}{(N-1)!} \leq MK^{N-1}\frac{(b-a)^N}{(N-1)!}$$

Since

$$\sum_{N=1}^{\infty} MK^{N-1}\frac{(b-a)^N}{(N-1)!} = M(b-a)e^{K(b-a)}$$

by Weierstrass M-test(Theorem 1.1.20) we conclude that the series $\sum_{N=1}^{\infty}[x_{i,N} - x_{i,N-1}]$ converges uniformly for all $z \in E$. But

$$x_{i,N}(z) = \sum_{k=1}^{N-1}\left[x_{i,k+1}(z) - x_{i,k}(z)\right] + x_{i,1}$$

Thus, the sequence $\{x_{i,N}\}_{N=1}^{\infty}$ converges uniformly to a function $x_i(z)$ for all $z \in E$ and hence the function $x_i(z)$ is a continuous function (Theorem 1.1.18). Also, we can interchange the order of taking limits and integration for such sequences. Therefore

$$\begin{aligned}
\mathbf{x}(z) &= \lim_{N\to\infty} \mathbf{x}_N(z) \\
&= \mathbf{x}_0 + \lim_{N\to\infty}\int_{z_0}^{z} (\mathbf{A}(s)\mathbf{x}_{N-1}(s))\,ds \\
&= \mathbf{x}_0 + \int_{z_0}^{z} \lim_{N\to\infty} (\mathbf{A}(s)\mathbf{x}_{N-1}(s))\,ds \\
&= \mathbf{x}_0 + \int_{z_0}^{z} \mathbf{A}(s)\mathbf{x}(s)ds
\end{aligned}$$

This shows that $x(z)$ is a solution to the integral equation **Eqn 2** and therefore a solution to **Eqn 1**.

Uniqueness

Now, the uniqueness of solution to **Eqn 2** follows from Gronwall's Inequality (Theorem 1.1.23). Suppose that $\mathbf{y}(z)$ and $\mathbf{r}(z)$ are two solutions to the initial value problem **Eqn 2**.
Let $E = \{(x+y\cdot i) \in \mathbb{C} : x \in [m,n],\ y \in [l,o]\}$ Then for all $z \in E$ we have

$$\|\mathbf{y}(z) - \mathbf{r}(z)\| \leq \int_{z_0}^{z} K\|\mathbf{y}(s) - \mathbf{r}(s)\|ds$$

Letting $u(z) = \|\mathbf{y}(z) - \mathbf{r}(z)\|$ we have

$$0 \leq \Re\{u(z)\} \leq \Re\{\int_{z_0}^{z} Ku(s)ds\}$$

so that by Gronwall's inequality by splitting the components into the real part with $C = 0$ and $h(z) = K$, we find $u(z) := 0$ in $[m,n] = \Re\{E\}$ and therefore $\Re\{\mathbf{y}(z)\} = \Re\{\mathbf{r}(z)\}$ for all $z \in \Re\{E\}$ and

$$0 \le \Im\{u(z)\} \le \Im\{\int_{z_0}^{z} Ku(s)ds\}$$

so that by Gronwall's inequality by splitting the components into the imaginary part with $C = 0$ and $h(z) = K$, we find $u(z) := 0$ in $[l,o] = \Im\{E\}$ and therefore $\Im\{\mathbf{y}(z)\} = \Im\{\mathbf{r}(z)\}$ for all $z \in \Im\{E\}$. Combining the above two results in $\mathbf{y}(\mathbf{z}) = \mathbf{r}(\mathbf{z})\ \forall z \in E$. This completes a proof of the Uniqueness for **Eqn 1**.

3.1.4 If Part

Now finally we have $\dim(\mathrm{Ker}(L)) = n = \dim(V_R^n)$.

Proof

Let L be defined as in Theorem 1.1.21.

Then Choose $z_0 \in I$. Define $T : \mathrm{Ker}(L) \to \mathbb{C}^n$ by

$$Ty := \left(y(z_0), y'(z_0), \ldots, y^{(n-1)}(z_0)\right)$$

As T is linear(Theorem 1.1.21) and then by uniqueness theorem, $T(y) = \mathbf{0}$ implies $y = 0$. Therefore, T is one-to-one. The existence of solution shows that T is onto. Thus, T is bijective. Hence $\dim(\mathrm{Ker}(L)) = n$ which is basically our solution space $(\dim(V_R^n))$.

3.1.5 Only If Part

$\Longrightarrow$ Now we come to the if part of the proof ie to prove that if $\dim(V_n^R) = n$ and $a_j\ , j \in \{0,1,2\cdots n-1\}$ are continuous in R then this implies that all the coefficient functions $a_j\ , j \in \{0,1,2\cdots n-1\}$ are all analytic in R.

Before proving it let me give an example for the case $n = 1$. Let us consider the homogeneous equation

$$y' = a(z) \cdot y$$

where $a(z)$ is continuous in R. Since our solution space is V_R^1 that means we have only 1 solution. Let this solution be $f(z)$ which is also a non-trivial solution of the above equation. If it was a trivial solution then there is nothing to consider as Constant functions are all analytic.

Now consider a point $z_0 \in R$ where $f(z_0) \neq 0$. Then we have the resulting equation

$$\frac{f'(z)}{f(z)} = a(z)$$

which is analytic in some neighbourhood of z_0 due to the fact that $\frac{f'(z)}{f(z)}$ is holomorphic in some neighbourhood of z_0 and since holomorphic implies analytic we get that $a(z)$

is analytic in some neighbourhood of z_0, and therefore $a(z)$ is analytic $\forall z \in R$ where $f(z) \neq 0$. Now by applying Theorem 1.1.10 and since $a(z)$ is continuous in R we get that $a(z)$ is analytic in R. Hence proved as an example for $n = 1$.

Now to prove the general case we proceed by induction.

Base Case

As stated by the example above that proves the base case when $n = 1$.

Inductive Hypothesis

Assume this holds true for some positive integer n ie

$$y^{(n)} = a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_0 y,$$

Induction

Now consider a homogeneous linear differential equation of order $n+1$.(**Eqn 1**)

$$y^{(n+1)} = a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y$$

where a_k ,$k \in \{0, 1 \cdots n\}$ are continuous functions in R.

Now let (V_R^{n+1}) be the solution set of the above equation with the dimension of it being $n+1$.

Let $y_1, y_2 \cdots y_{n+1}$ be the basis for this vector space of dimension $n+1$. Now choose $z_0 \in R$ such that $y_1(z_0) \neq 0$. Let us define D to be the neighbourhood of z_0 such that $y_1(z) \neq 0$, ,$z \in D$. Then consider the set of n functions in this manner

$$\{\frac{y_2}{y_1}, \frac{y_3}{y_1}, \cdots \frac{y_{n+1}}{y_1}\}$$

are all analytic in D as $y_1 z_0 \neq 0$.

Now define $Y_k = (\frac{y_k}{y_1})'$, $k = 2, 3 \cdots n+1$. Then $Y_2 \cdots Y_{n+1}$ are a set of n functions that are analytic in D.

Now we will show that $Y_2, Y_3 \cdots Y_{n+1}$ are all linearly independent in D.

Suppose $\sum_{k=2}^{n+1} c_k Y_k = 0$. Then

$$\left(\frac{\sum_{k=2}^{n+1} c_k y_k}{y_1}\right)' = 0$$

Thus, we must have the inside a constant function, say C. Then

$$\frac{\sum_{k=2}^{n+1} c_k y_k}{y_1} = C$$

This gives a linear relation

$$\sum_{k=2}^{n+1} c_k y_k - C y_1 = 0$$

Since $y_1, \ldots, y_{n+1}$ are linearly independent, we must have

$$C = c_2 = \cdots = c_{n+1} = 0$$

This shows that $Y_2, \ldots, Y_{n+1}$ are linearly independent in D.

This proves our claim and now we reduce the order of **Eqn 1** by 1 as we know the solution y_1 and then we get the fact that $\forall k \in \{2, 3 \cdots n+1\}$ Y_k is a solution for this reduced n'th order homogeneous equation as also done in the class we obtain ie **Eqn 2**

$$u^{(n)} = c_{n-1}u^{(n-1)} + c_{n-2}u^{(n-2)} + \cdots + c_0 u$$

in D with coefficients $c_{n-1}, c_{n-2}, \ldots, c_0$, continuous in D, where

$$c_{n-1} = y_1^{-1}\left[-\begin{pmatrix} n+1 \\ 1 \end{pmatrix} y_1' + a_n y_1\right]$$
$$c_{n-2} = y_1^{-1}\left[-\begin{pmatrix} n+1 \\ 2 \end{pmatrix} y_1'' + \begin{pmatrix} n \\ 1 \end{pmatrix} a_n y_1' + a_{n-1} y_1\right]$$
$$c_{n-3} = y_1^{-1}\left[-\begin{pmatrix} n+1 \\ 3 \end{pmatrix} y_1''' + \begin{pmatrix} n \\ 2 \end{pmatrix} a_n y_1'' + \begin{pmatrix} n-1 \\ 1 \end{pmatrix} a_{n-1} y_1' + a_{n-2} y_1\right]$$
$$c_0 = y_1^{-1}\left[-\begin{pmatrix} n+1 \\ n \end{pmatrix} y_1^{(n)} + \begin{pmatrix} n \\ n-1 \end{pmatrix} a_n y_1^{(n-1)} + \begin{pmatrix} n-1 \\ n-2 \end{pmatrix} a_{n-1} y_1^{(n-2)} + \cdots + a_1 y_1\right]$$

or in general $c_{n-k} = y_1^{-1}\left[-\begin{pmatrix} n+1 \\ k \end{pmatrix} y_1^k + \sum_{i=0}^{k-1} \begin{pmatrix} n-i \\ k-i \end{pmatrix} a_{n-i} y_1^{k-i}\right]$

Now let us set V_D^n to be the solution set of **Eqn 2** in D. Since we know that the set

$$\{Y_k :\ k \in \{2, 3 \cdots n+1\}\}$$

is a linearly independent set it implies that our solution space has dimension n ie $\dim(V_D^n) = n$.

Now applying the inductive hypothesis then we gain that each

$$\{a_k :\ k \in \{1, 2 \cdots n\}\}$$

in analytic in D and since by **Eqn 1** holds $\forall z \in D$ we get that a_0 is analytic in D , which implies

$$a \in \{a_k :\ k \in \{0, 1, 2 \cdots n\}\} \text{ is analytic at each point } z \in R \text{ such that } y_1(z) \neq 0$$

Then again since we already know that the zeros of y_1 are isolated (Theorem 1.1.18) and each a_k is continuous in R we get that each a_k is analytic in R.

Q.E.D

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