Problem Chosen	2022 MCM/ICM Team Solution	Team Control Number
B		2221823

Water and Hydroelectric Power Sharing

Abstract

Water is required for home and industrial purposes, irrigated agriculture, hydro power generation, and the proper functioning of an ecosystem function. Climate change and ever increasing demands for water consumption has lead to an international battle to control the flow of water. One such interesting dilemma is that of the Colorado River System that primarily passes through the five states (AZ, CA, WY, NM, and CO) in U.S and ends in Mexico. In this paper, we turn this dilemma into a flow network problem, equivalently into a Linear Program to meet the basic water needs of a state whilst providing Mexico's water share. We implemented our model with fail-safe cases in cases of a water shortage, and consequently redirect water from either the dams, Mexico or even individual states as the drought gets progressively worse. To conclude, our model is simple, intepretable, fast and powerful enough to incorporate all the additional constraints to safely allocate water in this region.

Keywords: LP, Network Flow, Water Allocation, Colorado Basin, MCM 2022

Contents

1 Introduction

Water availability is a major issue in the Western United States. A reliable source of high-quality water is critical for public and ecological health, community stability, and economic prosperity. In this paper we discuss the logistics behind the Colorado River System.[8]

Colorado River Basin [5]

Spanning parts of the states of Arizona, California, Colorado, New Mexico, Nevada, Utah and Wyoming, the Colorado River Basin is one of the most foundations sources of water in the West Coast.[8] The Colorado River is approximately 1,450 miles long, with headwaters in Colorado and Wyoming, and finally flows into Mexico.[8] The river and its tributaries supply water to roughly 40 million people both inside and beyond the basin, and it irrigates nearly 5.5 million acres of agricultural land. [8] They are managed by a vast network of dams, reservoirs, and aqueducts, which redirect the river's entire flow for agricultural irrigation and household water supply in most years.[8] Additionally some of its waters are held back to create reservoirs by constructing dams.[8] Due to its large flow and steep gradient the dams are used for generating hydroelectric power.[8]

The Hoover Dam and the Glen Canyon Dam are two such dams in the Colorado River Basin System that we discuss in our paper. Natural resource officials in the U.S. states of Arizona , California , Wyoming , New Mexico , and Colorado are currently negotiating to determine the best way to use these two dams to manage water usage and electricity production. The volume of water from sources that feed dams and reservoirs is diminishing in many locations as a result of climate change. As a result, dams may be unable to supply the water needs in these locations. Furthermore, low water flow reduces the quantity of energy generated by hydroelectric facilities, causing disruptions in the power supply. If the water level in the reservoir behind the dam falls below a certain level, hydroelectric power generating ceases.

In this paper we propose a simplistic flow model which can capture all of these constraints and satisfy the basic needs of the states. Several situations were also considered as explained in section 5 such as (a) the case where none of the basic requirements of the states are met and hence we propose an idea of redirecting the output to California mostly as it has the highest yield and how that could enable the collaboration between states.

2 Network Flows

According to [6] a flow network (also known as a transportation network) is a directed graph with each edge having a capacity and receiving a flow. The quantity of flow on an edge cannot exceed the edge's capacity. A directed graph is commonly referred to as a network in operations research, with the vertices referred to as nodes and the edges referred to as arcs. Unless a node is a source with only outgoing flow or a sink with just incoming flow, a flow must fulfill the limitation that the amount of flow into it equals the amount of flow out of it. A network can be used to simulate traffic in a computer network, circulation with demands, fluids in pipelines, and electrical currents.

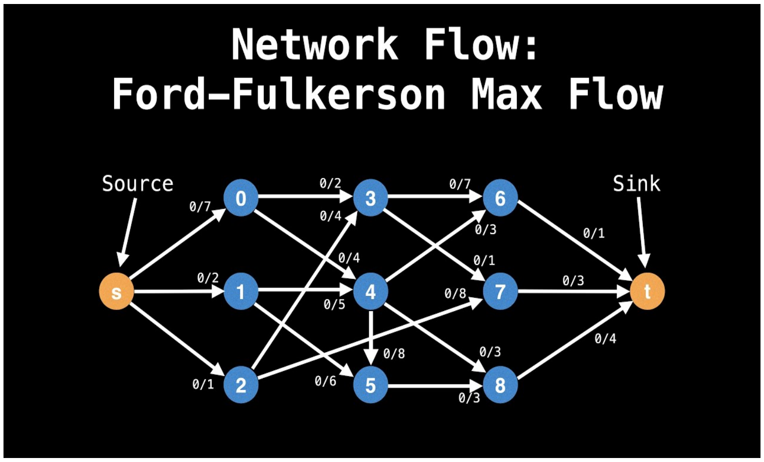


Figure 2: A network Flow

2.1 Formal Definition

Let $G = (V, E)$ be a directed graph where each $(u, v) \in E$ has a capacity $c(u, v) \geq 0$. Further, let $s \in V$ be a source vertex and $t \in V$ be a sink vertex. Here $s \neq t$.

An $s - t$ flow is an assignment of values to edges: $f : E \longrightarrow \mathbb{R}$ subject to:

- Non negative: $f(u, v) \geq 0$ for each $(u, v) \in E$.
- Capacity Respecting: $f(u, v) \leq c(u, v)$ for each $(u, v) \in E$
- Flow Conservation: $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,w)\in E} f(v, w)$ for each $v \in V - \{s, t\}$

If $S \subseteq E$ is a subset of edges, we let $f(S) := \sum_{(u,v)\in S} f(u, v)$. If v is a vertex, $\delta^{\text{in}}(v) := \{(u, x) \in E : x = v\}$ and $\delta^{\text{out}}(v) := \{(x, w) \in E : x = v\}$ are the edges entering and exiting v.

So the last constraint is equivalent to $f\left(\delta^{in}(v)\right) = f\left(\delta^{out}(v)\right)$ for each $v \in V - \{s, t\}$.
The value of a flow f is the net flow exiting s (i.e. the amount of flow being injected externally into s):

$$\text{val}(f) = f\left(\delta^{\text{out}}(s)\right) - f\left(\delta^{\text{in}}(s)\right)$$

2.1.1 Lemma 1

The value of the flow is equal to the total flow in the sink subtracted from the total flow in the source or more formally

$\operatorname{val}(f)=f\left(\delta^{in}(t)\right)-f\left(\delta^{\text{out}}(t)\right)$

2.2 Residual Network

Let $G_f=\left(V, E_f\right)$ be the following graph.

- Original Edge: For each $(u, v) \in E$ with $f(u, v)<c(u, v)$, include (u, v) in E_f with capacity $c_f(u, v):=c(u, v)-f(u, v)$.
- Reverse Edge: For each $(u, v) \in E$ with $f(u, v)>0$, include a brand new edge (v, u) with capacity $c_f(u, v):=f(u, v)$.

We call G_f the residual graph and c_f the residual capacities There may now be more than one edge from u to v : an original edge and a reverse edge. Regard these two edges as distinct.

2.2.1 Augmenting Path

For a flow f, an augmenting path is simply an $s-t$ path P in G_f. Let $c_f(P):=\min_{(u,v)\in P} c_f(u, v)>0$ be the minimum residual capacity of an edge on P. We will use P to increase val (f) by $c_f(P)$.

Let P be an augmenting path. We modify flow f to get another flow f' as follows:

- $f'(u, v)=f(u, v)+c_f(P)$ if the original edge (u, v) is on P.
- $f'(u, v)=f(u, v)-c_f(P)$ if the reverse edge (v, u) is on P.
- $f'(u, v)=f(u, v)$ otherwise.

We call this process augmenting f by P.

2.3 Max-Flow/Min-Cut

Let f be any flow such that there is no $s-t$ path in G_f. Let C be all nodes reachable from s in G_f. Then C is an $s-t$ cut with val $(f)=c\left(\delta^{\text{out}}(C)\right)$.

Proof

We have $s \in C$ because s is trivially reachable from s and $t \notin C$ by the assumption there is no $s-t$ path in G_f. Earlier we saw val $(f)=f\left(\delta^{\text{out}}(C)\right)-f\left(\delta^{\text{in}}(C)\right)$ But $f\left(\delta^{in}(C)\right)=0$, otherwise there would be an edge $(v, w) \in E_f$ with $v \in C, w \notin C$. This would contradict v being reachable yet w not being reachable from s in G_f.

Similarly, $f\left(\delta^{\text{out}}(C)\right)=c\left(\delta^{\text{out}}(C)\right)$ otherwise $f(u, v)<c(u, v)$ for some $(u, v) \in \delta^{\text{out}}(C)$ meaning $(u, v) \in E_f$. But, again, this is impossible since v is not reachable from s in G_f.

2.4 Ford - Fulkerson Algorithm

We present a very simple algorithm. Consider the following algorithm for network flow:

- $f \leftarrow$ the 0 flow (i.e. $f(u,v) = 0$ for each edge E)
- While there exists an $s-t$ path P in G_f
  - Augment f using this path P, now let f be this new flow
- Return f

This is the Ford-Fulkerson algorithm.

2.4.1 Integrality of Flows

If each $c(u,v)$ is an integer, then there is a maximum flow f with $f(u,v)$ being an integer for every edge. The Ford-Fulkerson algorithm finds an integer maximum flow in finite time.

2.5 Edmond's - Karp Algorithm

We present another algorithm that does in fact terminate in polynomial time. Consider the following algorithm for network flow:

- $f \leftarrow$ the 0 flow (i.e. $f(u,v) = 0$ for each edge E)
- While there exists a $s-t$ path P in G_f (choose the shortest $s-t$ path)
  - Augment f using this path P, now let f be this new flow
- Return f

This is the Edmond's - Karp algorithm which runs in $O(|V||E|^2)$ time.

3 Linear Programming

Given limited resources and competing restrictions, many issues take the form of maximizing or reducing an objective. We have a linear programming issue if we can express the aim as a linear function of some variables and the restrictions on resources as qualities or inequalities on those variables. Linear programming appear in a wide range of practical applications.

3.1 General Linear Programs

Following the textbook [2] .In the general linear-programming problem, we wish to optimize a linear function subject to a set of linear inequalities. Given a set of real numbers $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$, we define a linear function f on those variables by

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{j=1}^{n} a_jx_j$$

If b is a real number and f is a linear function, then the equation

$$f(x_1, x_2, \ldots, x_n) = b$$

is a linear equality and the inequalities

$$f(x_1, x_2, \ldots, x_n) \leq b$$

and

$$f(x_1, x_2, \ldots, x_n) \geq b$$

are linear inequalities. The phrase linear constraints is used to refer to either linear equalities or linear inequalities. We do not tolerate tight inequalities in linear programming. A linear-programming issue is formally defined as the problem of reducing or maximising a linear function under a limited set of linear constraints. If we want to minimise, we call the linear program a minimization linear program; if we want to maximise, we call it a maximisation linear program.

3.2 Flow Network as a Linear Program

As stated in CLRS [2] We express the maximum-flow problem as a linear program. Recall that we are given a directed graph $G = (V, E)$ in which each edge $(u, v) \in E$ has a non negative capacity $c(u, v) \geq 0$, and two distinguished vertices: a source s and a sink t.

As defined above , a flow is a non negative real-valued function $f : V \times V \to \mathbb{R}$ that satisfies the capacity constraint and flow conservation.

A maximum flow is one that meets these limits while also maximizing the flow value, which is the total flow out of the source minus the total flow into the source.

A flow, therefore, satisfies linear constraints, and the value of a flow is a linear function. Recalling also that we assume that $c(u, v) = 0$ if $(u, v) \notin E$ and that there are no anti parallel edges, we can express the maximum-flow problem as a linear program:

$$\max \sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$$

subject to

$$\begin{aligned} f_{uv} \leq c(u, v) &\text{ for each } u, v \in V, \\ \sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv} &\text{ for each } u \in V - \{s, t\}, \\ f_{uv} \geq 0 &\text{ for each } u, v \in V. \end{aligned}$$

Note that it is often easier to incorporate extra features in a linear program rather than in a graph.
There are various polynomial time algorithms which can be used to solve this and some of the further limitations of this approach specifically in our case our discussed in the drawback section.

4 Mathematical Model

We now discuss our mathematical model where we state what assumptions and biases we have.

4.1 Assumptions

We make the following set of of assumptions.

1. Each iteration which we is our unit of time is considered equivalent to a single day in our model.
2. We assume that there is no backtracking of water between the water that excess clean water that over the cities and the source. In other words, the source and the sink aren't connected as we don't want the cities to be sending the excess water back to the source but rather to Mexico.
3. The Source does not have an infinite amount of supply of water, but can be replenished by rainfall in a finite manner.
4. We feel that the basic water needs of Mexico is more important than the water for our hydroelectricity given that most states we consider the U.S mainly rely on other forms of energy by Table 1 and consequently can make use of them.
5. Each state is supplied with the water that meets its needs, no state has individual reservoirs.
6. The importance of water consumption in Industry is of less significance compared to Residential and Agricultural consumption. Moreover, we value the Residential consumption as our top priority to ensure that the states' populous does not experience a water crises.
7. Every state has the best interest of each other and not only of their own state. Each state will share their resources with each other in the case of a crisis such that the outcome leads to a minimization in damages during a crisis.

4.2 Graph Configuration

We begin developing our model by first creating our source and the sink. The source in our model is the Colorado River system. The outflow of water is then sent to the Glen Canyon dam (Lake Powell) and the Hoover dam (Lake Mead) with certain maximum value M and P respectively. Note that without Loss of generality we convert the water level to the volume of water in both the dams. Since water outflows from the Glen Canyon dam supply part of the water input to the Hoover dam we mark that constraint by T. We set it to 10% of P and this can value can change depending on the circumstances like droughts, climate changes etc.

Now the water out flowing from both the lakes we constraint it by H' and H'' as we set a hard constraint that we need at least H' and H'' amount of water for hydroelectricity. According to studies, we need at least 3,490 feet of water(water level) and for Lake Powell and 940 feet of water (water level) needed for Lake Mead. This is stated in [3] If water levels drop below 3,490 feet, the so-called minimum power pool, the Glen Canyon Dam, which supplies electricity for about 5.8 million customers in the inland West, will no longer be able to generate electricity. Hence we set a hard capacity of around 50%M and 50%P for both the dams to ensure this constraint.

Now lastly we make a set of assumptions based on the observations from the data related to the 5 States.

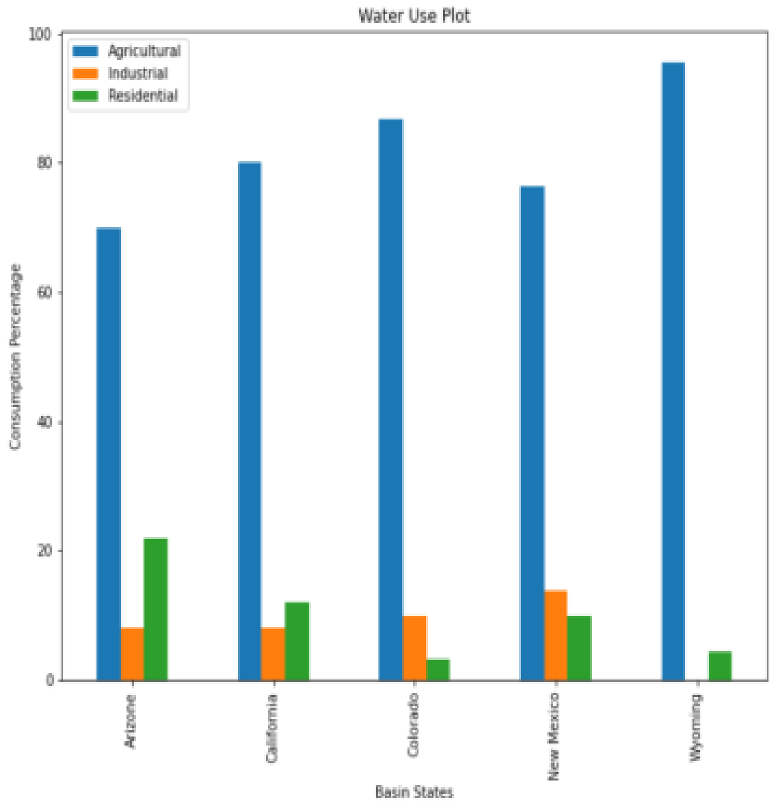


Figure 3: Water consumption rates for the 5 States

4.2.1 Observations - 5 States

According to the data plotted in Figure 3.

- All states use most of the water for Agriculture
- Arizona and California use more water in residences rather than in the Industry and the remaining 3 states use more water for their Industry.
- We will assign a total ordering of our S_i $i \in \{1, 2, 3, 4, 5\}$ where we assume that the ordering comes from the requirement for Agriculture dominates and subsequently the next ordering comes from Residential consumption and lastly Industry.
- Hence $S_2 > S_5 > S_3 > S_4 > S_1$

Mexico has claims on the residual water left after the five states have consumed their shares. Hence whatever(if any) is the residual from these 5 states is then given to Mexico.

The final Network flow is as follows on page 10, where the edges are marked as in Table 2 on page 11. The column Dynamic Constraint just implies that depending on the situation the capacity on the edges can change its value (hence this is a dynamic and not a static flow).

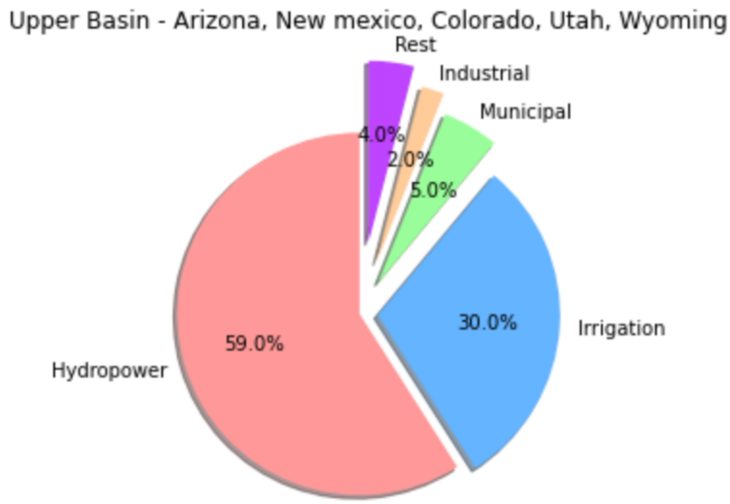


Figure 4: Graph showing Water consumption rates for the Upper Colorado River Basin

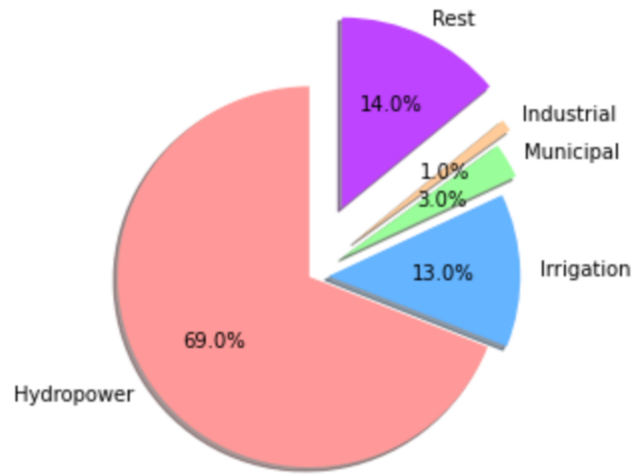
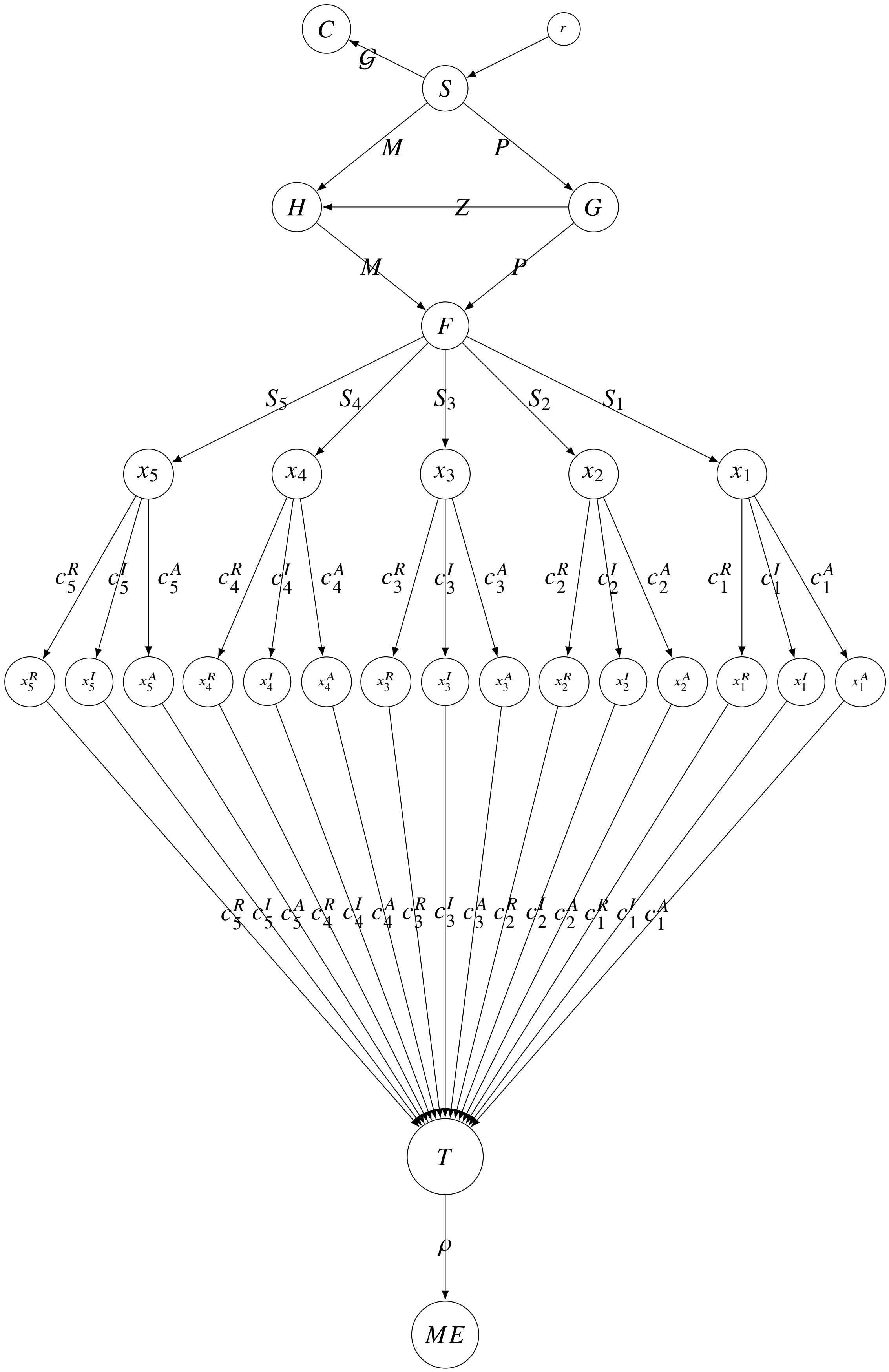


Figure 5: Graph showing Water consumption rates for the Lower Colorado River Basin

Table 1: Historical Data (2017) [4]

State	Agricultural Output (% of U.S. output)	Electricity generated With Hydro power (% of Total Electricity)	Electricity generated Without Hydro power (% of Total Electricity)
Arizona	1.36	6	94
California	13.52	20	80
Colorado	2.02	3	97
New Mexico	0.86	<1	>99
Wyoming	0.41	3	97



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T
ρ
ME

Table 2: Variables used in our model

Variable	Description	Unit	Value	Dynamic Constraint
M	Water level in Lake Mead (converted to volume of water)	m^3	-	Yes
P	Water level in Lake Powell (converted to volume of water)	m^3	-	Yes
Z	Amount of water outflowing from the Glen Canyon Dam to the Hoover Dam	m^3	10%P	Yes
H'	Water level reserved for Hydroelectric power for Lake Mead	m^3	50%M	No
H"	Water level reserved for Hydroelectric power for Lake Powell	m^3	50%P	No
S_1	Total water required for Arizona	m^3	-	Yes
S_2	Total water required for California	m^3	-	Yes
S_3	Total water required for Wyoming	m^3	-	Yes
S_4	Total water required for New Mexico	m^3	-	Yes
S_5	Total water required for Colorado	m^3	-	Yes
$\mathcal{G}$	Volume of water required for the Gulf of California	m^3	-	Yes
c_1^A	Amount of water required by Arizona for Agricultural Purposes	m^3	50%S_1	Yes
c_1^I	Amount of water required by Arizona for Industrial Purposes	m^3	25%S_1	Yes
c_1^R	Amount of water required by Arizona for Residential Purposes	m^3	25%S_1	Yes
c_2^A	Amount of water required by California for Agricultural Purposes	m^3	50%S_2	Yes
c_2^I	Amount of water required by California for Industrial Purposes	m^3	25%S_2	Yes
c_2^R	Amount of water required by California for Residential Purposes	m^3	25%S_2	Yes
c_3^A	Amount of water required by Wyoming for Agricultural Purposes	m^3	50%S_3	Yes
c_3^I	Amount of water required by Wyoming for Industrial Purposes	m^3	25%S_3	Yes
c_3^R	Amount of water required by Wyoming for Residential Purposes	m^3	25%S_3	Yes
c_4^A	Amount of water required by New Mexico for Agricultural Purposes	m^3	50%S_4	Yes
c_4^I	Amount of water required by New Mexico for Industrial Purposes	m^3	25%$S_4 S_4$	Yes
c_4^R	Amount of water required by New Mexico for Residential Purposes	m^3	25%S_4	Yes
c_5^A	Amount of water required by Colorado for Agricultural Purposes	m^3	50%S_5	Yes
c_5^I	Amount of water required by Colorado for Industrial Purposes	m^3	25%S_5	Yes
c_5^R	Amount of water required by Colorado for Residential Purposes	m^3	25%S_5	Yes
y_1^A	Minimum amount of water required by Arizona for Agricultural Purposes	m^3	-	No
y_1^I	Minimum amount of water required by Arizona for Industrial Purposes	m^3	-	No
y_1^R	Minimum amount of water required by Arizona for Residential Purposes	m^3	-	No
y_2^A	Minimum amount of water required by California for Agricultural Purposes	m^3	-	No
y_2^I	Minimum amount of water required by California for Industrial Purposes	m^3	-	No
y_2^R	Minimum amount of water required by California for Residential Purposes	m^3	-	No
y_3^A	Minimum amount of water required by Wyoming for Agricultural Purposes	m^3	-	No
y_3^I	Minimum amount of water required by Wyoming for Industrial Purposes	m^3	-	No
y_3^R	Minimum amount of water required by Wyoming for Residential Purposes	m^3	-	No
y_4^A	Minimum amount of water required by New Mexico for Agricultural Purposes	m^3	-	No
y_4^I	Minimum amount of water required by New Mexico for Industrial Purposes	m^3	-	No
y_4^R	Minimum amount of water required by New Mexico for Residential Purposes	m^3	-	No
y_5^A	Minimum amount of water required by Colorado for Agricultural Purposes	m^3	-	No
y_5^I	Minimum amount of water required by Colorado for Industrial Purposes	m^3	-	No
y_5^R	Minimum amount of water required by Colorado for Residential Purposes	m^3	-	No
y^{ME}	Minimum amount of water required by Mexico for all purposes	m^3	-	No
ρ	Residual water for Mexico	m^3	15%T	Yes
r	Rainfall	m^3	-	No

4.3 Our model as a Linear Program

The linear programming model's goal function to be applied to the established network architecture is the one developed in the previous section.

Let $V = \{S, C, G, H, F, x_1, x_2, x_3, x_4, x_5\} \bigcup \{x_i^j\} \ \forall i \in \{1,2,3\} \ \forall j \in \{A, I, R\} \bigcup \{T, ME\}$ where x_1^A represents the minimum amount of water required for state 1 i.e Arizona for agricultural purpose. Let E defined as above in the graph and let us define $V' \subset V$ such that $V' = \{x_i^j\} \ \forall i \in \{1,2,3\} \ \forall j \in \{A, I, R\}\} \bigcup \{T\}$

Then let our **objective** function be defined as follows(We subtract the last node ME due to the fact that we just want to maximize the amount of water flowing to the node T and then we assign 15% of it to Mexico as the residual water).

$$\max \sum_{v \in V - \{ME\}} f_{sv} - \sum_{v \in V - \{ME\}} f_{vs}$$

subject to

$$f_{uv} \le c(u, v) \ \forall u, v \in V,$$
$$\sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv} \ \forall u \in V - \{S, T\},$$
$$f_{uv} \ge 0 \ \forall u, v \ \in V.$$
$$0 \le P - H' \le f_{G,T} \le P$$
$$0 \le M - H'' \le f_{H,T} \le M$$
$$0 \le y_1^j \le f_{x_1, x_1^j} \le c_1^j \ \forall j \in \{A, I, R\}$$
$$0 \le y_2^j \le f_{x_2, x_1^j} \le c_2^j \ \forall j \in \{A, I, R\}$$
$$0 \le y_3^j \le f_{x_3, x_1^j} \le c_3^j \ \forall j \in \{A, I, R\}$$
$$0 \le y_4^j \le f_{x_4, x_1^j} \le c_4^j \ \forall j \in \{A, I, R\}$$
$$0 \le y_5^j \le f_{x_5, x_1^j} \le c_5^j \ \forall j \in \{A, I, R\}$$
$$0 \le y^{ME} \le f_{T,ME} \le 15\% \sum_{v \in V'} f_{uv} \ \ \forall u \in V'$$

In summary the constraints satisfy all the properties of the flow i.e non-negativity , conservation of flow and capacity constraints.

Next we have two additional constraints such that we satisfy the required property of reserving some amount of water for Hydroelectricity at the two dams which were discussed in detail in the section 4.1.

We also include additional constraints such that for each edge from $x_i \ \forall i \in \{1,2,3,4,5\}$ to $x_i^j \ \forall i \in \{1,2,3\} \ \forall j \in \{A, I, R\}$ (example : the edge (x_1, x_1^A)) we satisfy the minimum requirement required by the state for whatever purpose it may be for ie Agriculture, Residential or Industrial.

The last constraint has the same property but also we assign at most 15% of the total water used by the states as residual water for Mexico.

5 Analysis

We do a case by case analysis of the model above. Specifically during two conditions - **Shortage** and **Surplus**.
Shortage.
(Since we are considering possible worst case scenarios we assume $r = 0$)

Case 1. When one/or more of the y_i^j fail. This means that when one or more of the sectors in each state/multiple states doesn't have enough water provided to operate even at their minimum requirement. In mathematical terms,

$$\exists y_i^j, c_i^j \ni c_i^j < y_i^j \iff F < \sum y_i^j \quad \forall y_i^j$$

Initially $S - (H' + H'') = F = \sum c_i^j \geq \sum y_i^j$. If $S < (H' + H'') + \sum y_i^j$, case one occurs. Also since in this case we have enough water $(H' + H'')$ to supply the states as well as Mexico's minimum requirements we can consider y^{ME} on the right sum by Assumption (4). If $(H' + H'')$ is arbitrarily small then we arrive at case 2. Therefore our objective function gets the following constraint:

$$S < (H' + H'') + \sum y_i^j + y^{ME} \quad \forall y_i^j$$

Proposed Solution:
Reduce H', H'' essentially diverting it to $F \implies$ providing the water required to operate for the respective y_i^js. This in turn reduces the water available for hydro power but by assumption (4) we give Mexico **only** the minimum amount of water for it to operate.

$$\rho = y^{ME}$$

Case 2. When

$$H' = H'' = 0$$

i.e when there's no water left to allocate for hydro-power. This is essentially case 1 without the $(H' + H'')$ term. However, in this case since we don't have enough water to provide

$$S < (H' + H'') + \sum y_i^j + y^{ME} \quad \forall y_i^j$$

$$\iff S < \sum y_i^j + y^{ME} \quad \forall y_i^j$$

Proposed solution:

$$\rho = 0$$

Since we don't have enough water for hydro power and we want to prioritize our states first. Thus, we decide to divert the water that was to be sent to Mexico to our states.

Case 3. When there's no water to send to Mexico, the amount of water in the source is less than minimum water required for states to operate and no water left for hydro power. In mathematical terms,

$$\rho = 0$$
$$S < \sum y_i^j \quad \forall y_i^j$$
$$H' = H'' = 0$$

From Table (1), due to a high agricultural output from California and Arizona we believe that the other states can benefit from the shared crops and during this state we need to prioritize water supply for residences as it is the basic necessity of life. Mathematically, (these are constraints and proposed solutions)

$$c_i^R \geq y_i^R \quad \forall y_i$$

$$c_2^A \geq y_2^A$$

$$c_1^A \geq y_1^A$$

Case 4. When any one of Case 3 conditions/constraints fail.
Proposed Solution: We maintain

$$c_i^R \geq y_i^R$$

i.e Water supply for residences is prioritize.

Surplus.
There's only one case in this scenario. This happens when

$$S - \mathcal{G} - (H' + H'') - \sum y_i^j - y^{ME} > 0$$

This means water demand for all 15 sectors of all states, Mexico and water required for hydro power are all satisfied and there's enough surplus water available to give to the the Gulf of California. (We also prioritise y^{ME} over $\mathcal{G}$ as we first want to make sure that every state and Mexico water demands are met first). To solve for $\mathcal{G}$,

$$S - (H' + H'') - \sum y_i^j - y^{ME} = \mathcal{G}$$

6 Results

We now run our model using the Edmond- Karp Algorithm by only considering a subset of the Nodes ie removing the nodes r, C, ME to ensure that the maximum flow occurs and we do not want any cycles or dead ends to occur. We mark the source node as 1 and the sink node as 25. We just generated a bunch of random numbers on our pipes and followed the same criteria as stated in section [4]. Note we can also simulate it using any Linear Programming Algorithm but we chose to stick with Graph Algorithms for intepretability and visualization.

In all our results lets fix the initial minimum requirements as these values to make the comparison simpler

- $M = 50$
- $P = 50$
- $y_i^j = 4 \ \forall i \in \{1, 2, 3\} \ \forall j \in \{A, I, R\}$

Case 1. When the source has adequate amount of water to satisfy all the needs of the states and even supply the residual to New Mexico and a bit of water to the Gulf of California.

Analysis We notice that the model is stable and converges and we get the maximum flow value of 100 when we initially supplied our source with 200 volumes of water. 100 of them were used for hydroelectricity and all the minimum demands for all the states were met. We not assign 15 volumes of water to Mexico according to our model and also 5 volumes of water can be assigned back to the Gulf of California from the Colorado River.

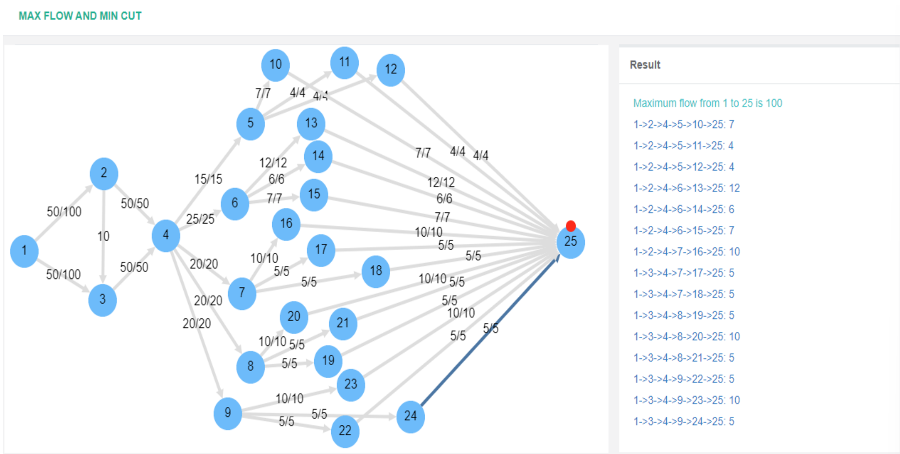

Figure 6: Maximum flow of our Model under Normal Conditions

Case 2. We consider case 2 of Shortage from the previous section ie we only have enough water for the states and do not have enough water to assign to Mexico so we assign no water and not enough water even for Hydroelectricity.

Analysis We notice that we do not assign any water to the Hydroelectric plant and only assign the whole 60 volumes of water to all the states (which are the bare minimum for each state (needing 12 volumes each). We do not assign the 15 volumes to Mexico or even have enough water for the Hydroelectric plants.

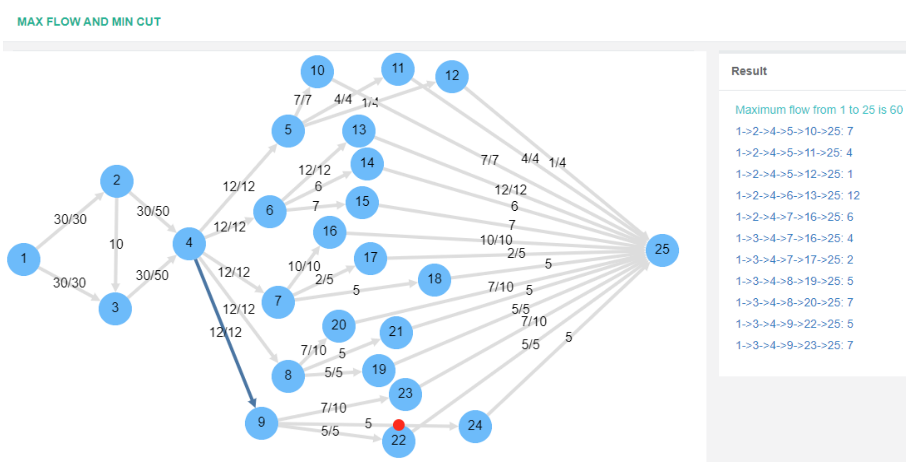

Figure 7: Maximum flow of our Model under Case 2 from Analysis

Case 3. When there's no water to send to Mexico, the amount of water in the source is less than minimum water required for states to operate and no water left for hydro power(we consider case 3 from Analysis)

Analysis We can see that the minimum requirement is 60 volumes of water overall but we fail to achieve that as we only have 50 volumes of water initially from the source. Therefore our model assigns the maximum value of the flow to California as we believe it has the highest yield (mentioned in the analysis section) which could help states such as New Mexico and Colorado and Wyoming. Our model does the same assignment as confirmed. Again as usual no water will be assigned to Mexico or the gulf of California as the basic needs of the states are more important.

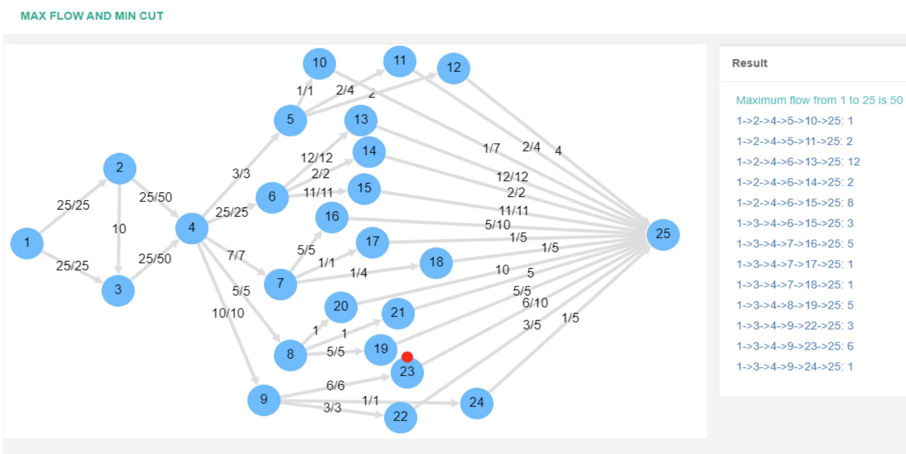

Figure 8: Maximum flow of our Model under Case 3 from Analysis

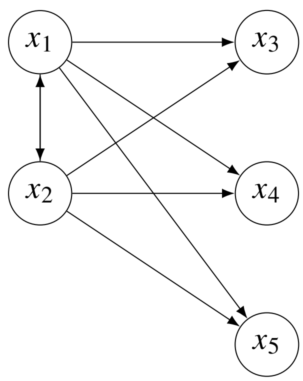

Figure 9: Relationship between California and Other states

We can think of the two states California and Arizona helping their fellow neighbouring states in this way. Figure 9 encapsulates the idea since California and Arizona get the most amount of water and they produce the highest yield they can share their resources with the other states.

Case 4. When we give Mexico only the minimum amount of water for it to operate.

Analysis Again our model assigns water to satisfy the minimum requirements of all the states and and only gives the minimum water required to Mexico and not more. We do this by diverting the water away from the Dams to satisfy the needs of the states.

Model Simulation Apart from this, we also provide code to simulate this which can be found in the appendix. Since we fix the measure of time to be 1 day in our flow network, by reasonably setting the parameters ie the source value to be 10000 and the dam levels to be 1000 and the model converges instantly and finds the exact maximum flow value which can be assigned.
We also simulated the case where it takes exactly 10 days for the source value to be equal to 0 and hence the model stops working at that point which rests our case that the model does incorporate and captures the issues well and assigns the best value to all the states. The code has 10 hyper parameters that can be modelled and changed to incorporate the various cases. All the parameters have been calibrated and we just chose a bunch of random values for simplicity and lack of time.

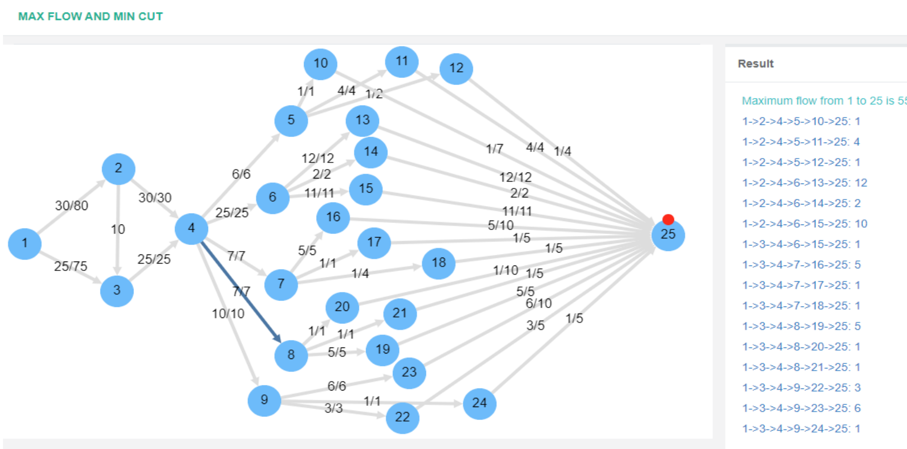

Figure 10: Maximum flow of our Model under Case 1 from Analysis

6.1 Validation Advantages

The validation of the model was conducted following the theoretical analysis done in section 5 where each flow value was checked to ensure the optimal value matches the theoretical best value. Since no data set was prepared to validate our model, we manually figured our the best values and compared it to our model's output. One of the advantages of our model is that it is simple and accurate enough to allocate water well to all the states and that the time complexity and space complexity of our model are both polynomial time. In fact our model is a Linear model which is advantageous when we have millions of parameters and further include more nodes.

6.2 Drawbacks

Even after optimization of our model, the allocation of water in real life could follow a different pattern and we did not compare to any real life data or utilize or rely on any existing historical agreements or current political powers of organizations or persons in these states.
Some of the limitations of our model

- We make a bunch of reasonable assumptions but in real life this could be different. It could be the case that sometimes Industry usage is prioritized over agriculture such as in places like California as they have a huge booming technology industry and if any shut down happens there the economy of the country could collapse.
- We could have implemented the concept of Dynamic Flow and not a static flow along with changing our unit of time
- By using Mixed Integer Linear Programming and including additional penalties we could have made our model more complex to simulate the real world better. An example would be to add a penalty every time the flow value does not meet the demand of a single y_i^j and hence the model would have to redirect the excess flow from other edges to this edges
- Compared to Physical based models which can include parameters for climate change and other demanding conditions our model is simple.

7 Conclusion

Our Linear Program-strength, model's however, is not dependent on the intricacy of its simulation of water flow, but on its ability to maximise the allocation of available water to the various states by satisfying the needs as well as making sure the best interest is in play. Finally to conclude our paper, we present the final discussion by answering all the questions. Note that most questions are already answered in detail in the previous sections

Question 1. *Recommend how often the model should be re-run to take into account changes in the supply and demand profiles.*
Proposed Solution: The model can be re-run depending on what parameters (the Lake Level's both M and P, source volume etc) are changed and an easy visualization tool [9] can also be used to gain more insight.

Question 2. *How much water does Mexico have a claim on after the five states have consumed their shares?.*
Proposed Solution: In our model Mexico is supplied with the minimum amount of water required (y^{ME}) for a day-to-day bases even if we go through a crisis wherein one/more state's minimum water requirement (y_i^j) if not satisfied.

Question 3. *After water allocations from your plan are implemented, discuss how much water (if any) should be allowed to flow into the Gulf of California from the Colorado River?*
Proposed Solution: During a surplus situation, we have allocated water to the Gulf of California and the amount allocated is just the residual water left after all the allocations are done. This is represented by the equation,

$$\mathcal{G} = S - (H' + H'') - \sum y_i^j - y^{ME}$$

Question 4. *When the water level in Lake Mead is M and the water level in Lake Powell is P, how much water should be drawn from each lake to meet stated demands?*
Proposed Solution: An amount sufficient to provide for all states minimum water requirement as well as for Mexico's water requirement provided the water required for hydro power isn't small. This can be represented mathematical by the following equations:

$$y_i^j + y^{ME}$$

$$H', H'' > 0$$

Question 5. *If no additional water is supplied (from rainfall, etc.), and considering the demands as fixed, how long will it take before the demands are not met?*
Proposed Solution: This depends on

- the proposed minimum requirements of each state y_i^j which is fixed.
- how much water is present in the initial source S.
- We can also re-iterate the code multiple times and take the average (each run initialises the code with different values for the capacities) which satisfy this constraint which would then give us an estimate of how long will it take before the demands are not met.

Question 6. *How much additional water must be supplied over time to ensure that these fixed demands are met?*
Proposed Solution: This depends on the same conditions as mentioned in Question 5. By changing the parameters in the model code ie (Change the source value and lake levels) we can estimate the number of days it could take to completely dry up ie the source has 0 volumes of water and then make a plan based on that. There is also another way, we know that the total amount of water at T should be greater than or equal to the sum of the minimum amount of water required by each state. If we are ever in case 2 of our analysis ,then we could always redirect water from the dams and provide the states.

Question 7. *What criteria you are using to resolve competing interests?*
Proposed Solution: During Normal conditions, all states sectors minimum water requirement y_i^j are full-filed. During a shortage, we prioritize California's and Arizona's water demand for agriculture as we believe they can help the other states by assumptions and also we make sure the residential minimum water requirements are met.

Question 8. *What should be done if there is not enough water to meet all water and electricity demands?*
Proposed Solution: This is the worst case scenario where during this state we try out best to provide for all residential water demands.

Question 9. *The demands for water and electricity in the communities of interest change over time. What happens when there is population, agricultural, and industrial growth or shrinkage in the affected areas?*
Proposed Solution: This is heavily discussed in our case study as well as one of our assumptions take care of this case.

Question 10. *What happens when the proportion of renewable energy technologies increases over the initial value used in your analysis?*
Proposed Solution: If more renewable energies are used, then we could allocate more water during droughts or shortages to all the cities rather than keeping any water for the hydroelectric generation.

Question 11. *What does your model indicate when additional water and electricity conservation measures are implemented?*
Proposed Solution: When additional water and electricity conservation measures are implemented, the total amount of required water and electricity decreases. This allows us to send more water to Mexico and if the change is significant enough, divert the water for hydroelectricity to the dams.

8 Appendix & Article

Code for the simulation of our model

```python
# source 11
from ortools.graph import pywrapgraph


def main():
    """MaxFlow simple interface example."""
    # Instantiate a SimpleMaxFlow solver.
    max_flow = pywrapgraph.SimpleMaxFlow()

    # Define three parallel arrays: start_nodes, end_nodes, and the capacities
    # between each pair. For instance, the arc from node 0 to node 1 has a
    # capacity of 20.
    start_initial = 10000
    while start_initial != 0:
        start_nodes = [1, 1, 2, 2, 3, 4, 4, 4, 4, 4, 5, 6, 7, 8, 9, 10]
        end_nodes =   [2, 3, 3, 4, 4, 5, 6, 7, 8, 9, 10, 10 , 10, 10, 10 , 11]
        capacity_res1 = 1000
        capacity_res2 = 1000
        cali = 350
        ari = 80
        wyo = 200
        new = 120
        colorado = 250
        capacities =  [start_initial,start_initial, int(start_initial*0.1), capacity_res1
                        ,capacity_res2, ari, cali, wyo, new, colorado, ari, cali, wyo, new, colorado, 150]

        # Add each arc.
        for arc in zip(start_nodes, end_nodes, capacities):
            max_flow.AddArcWithCapacity(arc[0], arc[1], arc[2])

        # Find the maximum flow between node 1 and node 11.
        status = max_flow.Solve(1, 10)
        if status != max_flow.OPTIMAL:
            print('There was an issue with the max flow input.')
            print(f'Status: {status}')
        print('Max flow:', max_flow.OptimalFlow())
        start_initial -= capacity_res1
        print('  Arc    Flow / Capacity')
        for i in range(max_flow.NumArcs()):
            print('%1s -> %1s   %3s  / %3s' %
                  (max_flow.Tail(i), max_flow.Head(i), max_flow.Flow(i),
                   max_flow.Capacity(i)))
if __name__ == '__main__':
    main()
```

Listing 1: Python example

Code for the plots.

```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd

# Bar Plot
plotdata = pd.DataFrame({
    "Agricultural":[70,80,86.7,76.3,95.6],
    "Industrial":[8,8,10,13.8,0.1],
    "Residential":[22,12,3.3,9.9,4.3]},
    index=["Arizone", "California", "Colorado", "New Mexico", "Wyoming"])
plotdata.plot(kind="bar",figsize=(10, 8))
plt.title("Water Use Plot")
plt.xlabel("Basin States")
plt.ylabel("Consumption Percentage")

# Pie chart 1
labels = ['Hydropower', 'Irrigation', 'Municipal', 'Industrial', 'Rest']
sizes = [59, 30, 5, 2,4]
explode = (0, 0.1, 0.2, 0.3,0.4)
colors = ['#ff9999','#66b3ff','#99ff99','#ffcc99','#BF3EFF']
fig1, ax1 = plt.subplots()
ax1.pie(sizes, explode=explode, labels=labels, colors=colors, autopct='%1.1f%%',
        shadow=True, startangle=90)
plt.title('Upper Basin - Arizona, New mexico, Colorado, Utah, Wyoming')
ax1.axis('equal')
plt.tight_layout()
plt.show()

# Pie chart 2
labels = ['Hydropower', 'Irrigation', 'Municipal', 'Industrial', 'Rest']
sizes = [69, 13, 3, 1, 14]
explode = (0, 0.1, 0.2, 0.3,0.4)
colors = ['#ff9999','#66b3ff','#99ff99','#ffcc99','#BF3EFF']
fig1, ax1 = plt.subplots()
ax1.pie(sizes, explode=explode, labels=labels, colors=colors, autopct='%1.1f%%',
        shadow=True, startangle=90)
plt.title('Lower Basin - Arizona, California, Nevada, New Mexico, Utah')
ax1.axis('equal')
plt.tight_layout()
plt.show()
```

Listing 2: Python example

The Tug O' Water between Mexico and the US

"Water, water everywhere, nor any drop to drink" is a phrase engrained in the minds of many young adults who have had a chance to read one of the longest major English poems "The Rime of the Ancient Mariner". The underlying tone behind the phrase is that of irony, the irony of being surrounded by ocean water, with not a drop to drink. However, with rapid climate change, and more frequent droughts , one can imagine a more dystopian scenario, that of being stranded in an ocean that once was.

A central geopolitical resource in the 21st century is water; whose demands exponentially increase by the year alongside its inherent value. It's paramount for countries to divert water sources emanating from their land to themselves as this is what's best for them; but water knows no human made boundaries. Rivers are welcomed with arms across international boundaries where they serve a crucial role to both the population and the economy. With water shortage cropping up over the horizon, the distribution of water is now a tug of war between neighbouring countries, which leads to rising tensions between them.

One such interstate and international struggle takes place between states of Arizona, California, Colorado, New Mexico and Wyoming in the US and Mexico. The central theme of the problem is that of optimization. From the outside, the problem, is merely *"Who gets how much?"*. But this is real life, and in real life, there are important nuances to take care of. One central theme is that of Mexico's water rights. Here is where geopolitical biases come into play. On one end, one could entirely leave Mexico out of the picture and divert our attention to the states. This approach would have humanitarian ramifications and would be a disaster for foreign policy. On the other hand, one could take a more humanitarian approach, where one prioritises basic human water and food needs over industry and energy, which however would quickly wreck havoc in economically well off states like California.

Figure 11: A picture of the Colorado Basin [10]

We believe that Mexico's water rights is as equally if not more important that the water for hydroelectricity given that the push for other non-hydro sources of energy in the states is on the rise. Some of these energy sources is green energy whereas others are in the form of fossil fuels. In the case of a water shortage, we believe in valuing Mexico's water requirements over our hydroelectricity. Certainly, hydroelectricity is one of the most cleanest sources of energy, and we are well aware that this cut would mean perhaps a more stronger dependence on non-renewable sources. However, we feel this is justified in the special case of a crisis.

Now that we have our biases stated, we now want to convey what we want to do into a model; this is mathematical modelling! A powerful method where we assign a bunch of symbols and define relations between them to construct a story. We start by associating the key players in our story as nodes belonging to a graph. Starting with the first node, i.e the Colorado river (the source) followed by the two dams (Hoover and Glen Canyon), the five states and finally the sink which is Mexico and the Gulf of Mexico all of which are individual nodes. This forms the skeleton of our model. Now, we tackle the problem of meeting each state's specific water demands. To do this, we can think of partitioning the water demands of an individual state into 3 main sections, that of Agriculture, Residence and Industry. This again adds more characters to our story, which are special nodes connected to each state. Finally, we connect each of these nodes with their respective partners and what we end up with is a graph (although incomplete).

Now once the nodes are connected, we add arrows to indicate the flow of water and a variable beside it to indicate the *amount of water* that moves between two nodes. We start by adding what we believe to be reasonable assumptions like non-negative flow (i.e the flow is only one way and only has positive units of water) and invoking the conservation law; the total water that flows into a node is the sum total of the water that is used by the node (if any) and the water the flows out. With our basic assumptions in mind, we add then the constraints of making sure that enough water is stored at the dams for hydroelectricity, enough water is sent through each states such that their demands are met. Finally, we also addressing Mexico's water requirements by ensuring its required water is sent. This simple model runs smoothly as long as the amount of water in the source is adequate to the states' individual and electrical demands alongside Mexico's water demands. During the case of water shortage or even worse, a drought, we need to divert according to our assumptions.

Intuitively, a drought is the worst case of a water shortage. To implement this mathematically, we need to be precise as to what we mean by a *water shortage*. We say we are in a *water shortage* when the amount of water flowing to one of the states fails meet its minimum demands for either Agriculture, Residence or Industry. This could happen either due to a rainfall shortage or the rapid drying of water bodies due to climate change. In these cases, we assume that the allocated water to the Gulf of California is nill. In each of the cases, our main priority is to ensure that the residential water demands are *always met*.

We now enter into **Phase 1**, which is the start of a water shortage. Since we still value Mexico's water rights, we decide to not restrict the flow towards Mexico, but rather reduce the water sent for hydroelectricity. This is a choice we decided to take as all of the states have other forms of non-hydroelectricity. However, we can only reduce the water allocated for hydro so much until it depletes and then we enter into the next phase.

In **Phase 2**, we are forced to meet with a choice between our states or Mexico. By the premise, we cannot let our states fail to meet its demands, and consequently we shut off the supply to Mexico and redirect it to our states.

Phase 3 starts when the amount of water from the source itself fails to meet any of our demands. Here our problem essentially breaks down into a local graph, where the nodes are each state's water supply. Knowing that California historically has the highest agriculture yield and consequently, we assume it has the biggest investment in agricultural technology, we set our model to divert all of the other states' industry water to California's agriculture sector in order to ensure a consistent food supply.

With our model now fully complete, including the fail-safe cases, we ran our model and found that our model indeed is able to optimize the allocation of the available water to all the states satisfying all the constraints as well as is able to successfully optimize the water allocation during droughts and other circumstances. The model does run in polynomial time and also instantly computes the maximum flow value that is possible. To conclude, our model is *simple*, *intepretable*, *fast* and *powerful* enough to incorporate all the additional constraints to safely allocate water in this region.

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